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On Time-to-Build Economies
with Multiple-Stage Investments

Introduction

Since the seminal paper by Kydland and Prescott [1982], dynamic general equilibrium models have been used to study the fluctuations of economic aggregates via the real business cycle theory, where exogenous productivity shocks are the principal factor behind changes in aggregated output. For further discussion see King and Rebelo [2000], Romer [2005], Wickens [2008].

One of the aspects of the model developed by Kydland and Prescott was that investments in future capital were time consuming and lasted for several periods before capital became operational. In order to produce one unit of capital, an agent who started an investment at time t would simply have to wait for the capital to be formed. Since investments were irrevocable, consumers were in fact incurring a sunk cost while making an investment, which did not produce any output until the formation of capital was complete. In fact, since the unfinished capital could not be used in production, consumers could not respond elastically to exogenous shocks, as their ability to smooth consumption inter-temporarily was significantly constrained by the capital formation technology.

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Though important, the approach by Kydland and Prescott presented the issue of time consuming capital formation in a simplified manner, as the transmission of an unfinished investment between stages was independent of the decision of an agent. At the start of an investment consumers determined the volume of capital which would eventually be formed, but had no impact on the duration of the process. Investments once started could not be stopped, postponed or finished earlier, which means the time of their completion did not vary endogenously due to exogenous shocks.

The abovementioned framework seems to be incomplete for two reasons. First of all, investments are usually an inter-temporal process in which each stage has to be financed separately. It is very unlikely that investors would be willing to pay for an investment once, at its start, rather than smooth the expenses over several periods. Whether in the housing market, or in production, the cost of investment is not incurred once, but is spread over a number of periods, each determined by a certain technological stage. What is more, due to technological issues, many expenses have to be covered sequentially, according to the specification of a given project. This implies that agents undertaking an investment have much more elasticity in allocating their resources among periods, rather than incurring a single sunk cost in the first stage of the capital formation process.

The other argument motivating this author to expand the approach by Kydland and Prescott is that the aforementioned elasticity of resource allocation allows agents to postpone or accelerate an investment by either reducing or increasing resources destined for a given project. This offers an opportunity to stop an unfinished investment when a negative economic shock hits an aggregated output, reducing the level of disposable income, and restart it once the economy has returned to its balanced path. This process might play a key role in an economic downturn, when agents are prone to postpone investments that have already been started in order to save scarce resources. Once the adverse economic situation is over, agents may be willing to recommence and complete previous investments. Such a situation might take place in housing markets, where long-lasting investments might be postponed and accelerated a number of times depending on the overall economic situation.

The two above arguments suggest that by enabling agents to decide on the level of multi-stage investments, as well as on the speed of their completion, one could describe economic dynamics more precisely and find a better explanation for developments taking place during an economic cycle.

From the technical point of view, introducing a model with the possibility of investing separately at different stages of a project, is directly related to several important issues. Throughout the paper a substantial number of our proofs will refer to dynamic programming arguments. In our work we will assume that it takes at least period J to complete one unit of capital. Each period t, the state variable is a measure of operating capital k_t, as well as a measure of unfinished projects, denoted as $s_{j,t}$, $j = 1, ..., J$, j-periods before

completion[1]. Usually in the literature, in order to prove the existence of an optimal allocation in an economy with an infinite horizon, a state space of the model is bounded in order to bound the domain of the corresponding value function, and thus bounding the function itself. This operation makes several dynamic programming arguments valid (see [Blackwell, 1965]).

It is quite common for different authors to impose an assumption on both the technology in the economy and the rate of capital depreciation, such that the space of values of capital could be bounded[2]. However, once we introduce to a model an economic variable that is not subject to depreciation, we might lose the boundedness of the corresponding state variable set, which makes the standard dynamic programming arguments invalid (see [Stockey, Lucas and Prescott, 1989]).

When analyzing a state variable of unfinished projects, one cannot exclude a situation when the measure of unfinished projects at a given stage increases in an unbounded manner. Even if one attempted to bound the state space by some $\bar{s}_m$, i.e. a level of unfinished projects that cannot be exceeded, such an operation would not find a reasonable economic interpretation. Even if not optimal, infinite measures of unfinished projects have to be taken into account when discussing time-to-build economies. Moreover, in order to bound the state space of unfinished projects, one should impose strong concavity assumptions on the investment technology, which in the case of our model is linear.

Even though the issue of unbounded returns arises naturally in economic models, they are not easy to deal with. When the instantaneous return function is continuous on its domain but unbounded, the Bellman equation constructed for a given optimization problem might not have a solution, as the corresponding operator might not have a fixed point (not to mention the lack of uniqueness of a fixed point, as the space of unbounded continuous functions is a Banach space only in very specific topologies). Since the space of unbounded continuous functions incorporated with the sup norm is not a compact space (nor is it a Banach space), the existing fixed-point arguments are invalid. This in particular concerns the techniques presented in Bertsekas and Shreve [1978] or Stockey, Lucas and Prescott [1989], which are based on the Banach contraction theorem and on Blackwell's sufficient conditions for contraction.

The above problem may occur in dynamic models due to the unbounded state space in which the value function is defined. As mentioned before, in some cases, the domain can be bounded in a natural way, so that the problem can be solved using the standard methods for bounded functions without affecting the economic value of the model (several ideas are presented in Stockey, Lucas and Prescott [1989]); however, in a great deal of examples, including ours, different techniques have to be used.

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[1]   In our work we use the notation presented in the paper by Kydland and Prescott [1982].

[2]   Usually this assumption can be reduced to the existence of some $\bar{k} < \infty$, such that $f(\bar{k}) = \delta\bar{k}$, where f denotes a technology in a given economy, and δ the rate of capital depreciation. In other words, this assumption states that there always exists a maximal level of sustainable capital, and thus the state space is reduced to $[0, \bar{k}]$.

Recent literature has widely focused on the idea of incorporating the function space with the so-called „weighted norms," or determining limit solutions for „truncated models" (see [Durán, 2003], [Stockey, Lucas and Prescott, 1989], and [Hernández-Lerma and Lasserre, 1999]). Some other important results, representing a different approach to the problem at various levels of generality, are presented in Le Van and Morhaim [2002], Le Van and Vailakis [2005], Le Van, Morhaim and Wailakis [2008], and Boyd [1990].

A very resourceful idea was presented by Rincón-Zapatero and Palmero [2003, 2007, 2009], who applied the so-called k-local contraction to study stochastic dynamic models with unbounded returns. Their framework was substantially improved by Matkowski and Nowak [2009], who proved the existence of a unique solution to the Bellman equation in a space of a broad class of unbounded functions. The main idea behind their work was to operate on a space of functions that are bounded in a given class of metrics on any bounded set. By showing that the Bellman equation had a unique solution on any of such bounded sets, it was sufficient to find a metric and a sequence of bounded sets, which preserve the property in the limit.

The second issue that arises when modelling time-to-build economies with multi-stage investments is the question of concavity and differentiability of the value function. In our framework, in order to move a unit of investment from one stage to another, the maximal level of resources destined for that purpose depends directly on the mass of projects in a given stage, since an agent can move no more than $s_{j,t}$ projects from stage j to $j + 1$ at time t. This implies that in many situations the optimal investment level might be boundary, as an agent might be willing to move exactly s_j projects to the next stage. To be precise, such a situation takes place in the steady state of our model, as agents continue to move the same level of investments throughout each period, in order to sustain the level of steady state capital, and invest exactly $s_{m,t}$. Since the standard argument in the literature by Benveniste and Scheinkman [1979] requires the optimal policies to be in the interior of the feasible action set, their result fails to hold in our framework. What is more, due to the specification of our model, the Inada conditions, which usually help rule out boundary solutions, also fail.

This may also be related to the question of the existence of prices in an economy, where the value function is either non-concave or non-differentiable, leaving aside computational issues[3]. The results concerned with the decentralization of non-convex and non-smooth models are based on a similar idea. Guesnerie [1975], in his pioneering study, based his proof on the convexity of the tangential approximations of production and preference sets, which were separated using the standard separation theorem. Other authors, including Brown [1991], Cornet [1981], and Bonisseau and Cornet [1988], exploited the properties of the Clarke tangent cones, convex by definition,

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[3] Notably, due to the presence of boundary solutions, optimal allocations and prices cannot be studied via the first-order conditions.

which make it possible to formalize marginal prices using the dual Clarke normal cone. Unfortunately, Clarke's normal cones may often be too large for an adequate description of marginal pricing, as shown in Khan [1988]. More satisfactory results for both finite and infinite-dimensional commodity spaces were obtained by Khan [1988, 1991, 1999] and Cornet [1990]. By applying non-smooth programming methods, they were able to formalize marginal prices by the basic normal cone. At the same time, the idea did not involve any sort of convex separation (for further discussion see chapter 8 in Mordukhovich [2006]).

The differentiability of the utility function in the optimal allocation is applicable in computing equilibrium prices; however, in many cases, smoothness is not obtainable. In dynamic programming literature, the differentiability of the value function is obtained mainly through the aforementioned theorem by Benveniste and Scheinkman [1979]. Unfortunately, the conditions stated in the theorem make it invalid when applying to many economic problems where boundary solutions arise naturally.

On the other hand, as shown by Rincón-Zapatero and Santos [2009], the interiority of the optimal solution is not necessary for the value function to be differentiable. In fact, they show that, under some constraint qualifications, the value function may remain differentiable, even though the optimal policy is on the boundary of the feasible action set.

Applying dynamic programming in the general equilibrium analysis is resourceful not only because one is able to prove the existence of prices supporting the equilibrium allocation, but also because it makes the proof constructive. The usage of Lagrange multipliers in welfare economics was probably presented for the first time by Lange [1942], although it was Negishi [1960] who showed their importance and economic interpretation. Once it was determined that the Lagrange multipliers of the social planner optimization problem are the equilibrium prices in the decentralized economy, the computation of equilibria became simpler, especially in smooth economies, where the Lagrange multipliers are equal to the marginal utility in the optimal allocation points. An analogous result was proven for the case of non-smooth convex economies by Papageorgiou [1960], where the equilibrium prices were the elements of the sub-differential of the utility function in the optimal allocation (for further discussion see Bewley [2007]). The advancements in the theory of dynamic programming, such as those concerning the infinite dimensional optimization problems (see [Le Van and Saglam, 2004], [Daniele, 2007], and [Daniele *et al.*, 2007]), or a non-smooth approach to the envelope theorem (see [Milgrom and Segal, 2002], [Tarafdar, 2009], and [Morand, Reffett and Tarafdar, 2009]), provide new tools for the analysis of the problem.

Even though the aforementioned arguments can guarantee that the value function of the corresponding optimization problem is differentiable, there is still the question of how to study optimal policies which might be boundary policies. Since they cannot be studied via first-order conditions imposed on the first-order derivatives of the corresponding Lagrangian, different methods

have to be implemented. What is more, even the standard monotone methods based on the supermodularity of the objective function are of little use, since the analyzed action sets, i.e. budget sets, are simplexes rather than lattices.

A standard approach to the theory of monotone comparative statics (MCS) has some limitations. A great deal of results presented in the literature cited above holds when the underlying set of actions or strategies is a lattice, unlike in the case of feasible action sets induced by budget constraints. This means that the MCS cannot be applied even to fundamental topics in economic theory[4].

A different approach was presented by Quah [2007]. By incorporating action sets with the so-called flexible set order, he showed that, under a weak notion of concavity of the objective function, optimal policy correspondences may be ascending in the standard interval order. His work finds great application in issues that could not be investigated using the standard MCS methods, including the theory of consumer choice.

In our model of a time-to-build economy, a representative consumer is maximizing his lifetime utility dependent on his consumption and leisure. A single consumption good is produced using a constant returns-to-scale technology, where the only two factors of production are capital and labor. Since we analyze a time-to-build economy, it takes several periods to form a single unit of capital. Unlike in Kydland and Prescott [1982], where an investment takes place only once during the whole process of capital formation, in our model an investment has to be made at each stage of its construction. Moreover, our economy is affected in each period by Markov productivity shocks.

The analysis of the model concerns several issues, which cannot be tackled using the standard methods for dynamic general equilibrium models. First, the state space of the economy is not bounded, and so is the corresponding value function. As mentioned before, this poses the question of the existence of a solution to the corresponding Bellman equation. Second, the law of motion imposed on state variables is defined using non-differentiable functions, which could compromise the smoothness of the value function itself. Moreover, once the law of motion of the state variables is not smooth, optimal policies cannot be studied via first-order conditions of the optimization problems. In addition, the budget set of the consumer is not a lattice, which makes the standard MCS arguments invalid.

In our work we provide three basic results. First, exploiting the work of Matkowski and Nowak [2009], we prove the existence of a unique unbounded value function satisfying the corresponding Bellman equation. Second, we characterize its monotonicity and strict concavity with respect to initial conditions. In addition, due to the work by Rincón-Zapatero and Santos [2009], we prove that the value function is continuously differentiable at every point in its domain. Eventually, profiting from the work by Quah [2007], we obtain

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[4]   Several ideas on how to deal with this particular issue using the MCS were presented in Antoniadou [2006] and Mirman and Ruble [2003]; however, they are based on orders that are of limited economic value.

the continuity of the optimal policy functions, as well as their monotonicity with respect to state variables. Eventually, we provide a constructive proof of the existence of prices supporting the optimal allocation, using the Negishi [1960] method.

In addition to the main results, we discuss the optimality of the presented economy. Moreover, we present a method for computing the value function, as well as equilibrium prices and allocations, and derive the theoretical error bounds.

Our paper is organized as follows. In the next paragraph we describe an economy, with the presentation of assumptions concerning preferences and technology. Next we proceed to analyze the model from the point of view of a social planner. In that section we provide our main results, concerning the existence and main properties of the value function, as well as the characterization of the optimal policies, and the steady state of the economy. In the following section we concentrate on an example of a decentralized economy and prove the two welfare theorems for our framework.

The model

In this section we present a model of a time-to-build economy with multiple-stage investments. First we present the assumptions with a suitable economic interpretation to which we will refer throughout the paper. Then we will move to the first part of the analysis, where socially optimal allocations will be characterized. We proceed with a description of our model.

Assumptions

In our economy we will consider a representative consumer whose instantaneous utility depends on consumption $c \in X$ and leisure $n \in N$. His instantaneous preferences over a set of feasible bundles of consumption and leisure are described by utility function u. The consumer is in possession of capital, denoted by k, which is used directly in the process of production. In addition, the agent may devote a part of his time to labor, denoted by l. A single consumption good is produced each period with technology f, dependent on k, l, and exogenous shock z. The shock values are governed by stochastic transition kernel Q. A product obtained each period can be either consumed or invested in future capital. Throughout this paper we will refer to the following assumptions. In order to clarify the notation, if $\phi(\cdot)$ is a real function, then $\phi_i(\cdot)$ will denote its first-order derivative with respect to variable i.

Assumption 1 (Preferences) *Let $c \in X \equiv \mathbf{R}_+$, $N \equiv [0, 1]$, and $u: X \times N \rightarrow \mathbf{R}$ be non-decreasing on $X \times N$. Moreover,*

    (i) *$u \in \mathcal{C}^1(X \times N)$,*
    (ii) *u is strictly concave on $X \times N$,*
    (iii) *$\forall c \in X, n \in N, u_c(0, n) = u_n(c, 0) = +\infty$.*

The above conditions are quite standard in research reports on dynamic general equilibrium models. First, we narrow our analysis to preferences over consumption and leisure, which can be represented by a differentiable utility function. Second, we expect the utility function to exhibit diminishing marginal utility with respect to both analyzed goods. Finally, we impose Inada conditions in order to rule out boundary solutions of consumer utility optimization.

Next we proceed with a description of production technology in the economy.

Assumption 2 (Technology) *Let $k \in K \equiv \mathbf{R}_+$, $z \in Z$, and $f: K \times N \times Z \to \mathbf{R}_+$ be non-decreasing on $K \times N$ for every $z \in Z$. Moreover,*

(i) *$f \in \mathcal{C}^1 (X \times N)$, $\forall z \in Z$,*
(ii) *f is concave on $X \times N$, $\forall z \in Z$,*
(iii) *$f (0,\cdot,\cdot) \equiv f (\cdot,0,\cdot) \equiv \mathbf{0}$, $\forall z \in Z$,*
(iv) *$\forall (k, l) \in K \times N$, $f (k, l,\cdot)$ is bounded,*
(v) *f is homogeneous of degree one.*

Again, the conditions presented in assumption 2 are standard for this line of literature. We assume that the technology is concave, i.e. exhibits decreasing marginal productivity, non-decreasing in the level of capital and labor, with constant returns to scale. Moreover, in order to produce strictly positive levels of output, both factors of production have to be strictly positive. The differentiability assumption is technical, however it implies smoothness of the economy.

Since in our description exogenous shocks play a crucial role, it is necessary to characterize a distribution from which shock values are drawn. We present a formal description with the following assumption.

Assumption 3 (Shocks) *Let $Z \subset \mathbf{R}$ be a compact set, and $\mathcal{Z}$ be its Borel set. Q is a transition function on $(Z, \mathcal{Z})$ satisfying the Feller property, i.e. for any measurable function $f: X \times Z \to R$, continuous on X, $h (\cdot, z) := \int_Z f (\cdot, z') dQ (z'|z)$ is continuous on X.*

Once we have presented the premises of the model, we proceed to the characteristics of the time-to-build technology, and the process during which the capital is formed.

In the time-to-build economy, it takes at least J periods to build one unit of new capital, i.e. a project started at time t will be completed no sooner than at time $t + J$. Let m denote the number of stages before a unit of capital is fully operational. In order to accomplish the construction of one unit of capital in J periods, it is necessary to invest $1/\lambda_m$ in every m^{th} stage, each $(t + J - m)^{th}$ period, $m = 1, ..., J$, $\lambda_m \geq 1$ and $\sum_{m=1}^{J} 1/\lambda_m = 1$. Let $\{1, ..., J\} \equiv \mathcal{J}$.

A measure of all projects which at time t are m stages before completion will be denoted by $s_{m,t} \in S_m \subseteq \mathbf{R}_+$. By investing $i_{m,t}$ at time t, an agent may

move a measure of $\lambda_m i_{m,t}$ projects from $s_{m,t}$ to a subsequent stage. However, it takes one whole period to complete one stage of investment. The law of motion of a measure of all unfinished projects m stages before completion can be described in the following way. $\forall m \in \mathcal{J}$,

$$s_{m,t+1} = \max\{s_{m,t} - \lambda_m i_{m,t}; 0\} + \min\{s_{m+1,t}; \lambda_{m+1} i_{m+1,t}\}. \tag{1}$$

According to (1), the evolution of $s_{m,t}$ is as follows. The measure of projects m periods before completion increases by the measure of projects moved from state $s_{m+1,t}$ to $s_{m,t}$, i.e. by $\lambda_{m+1} i_{m+1,t}$, but no more than the measure of projects at stage $s_{m+1,t}$. Analogously, $s_{m,t}$ decreases by the measure of projects which have been moved from state $s_{m,t}$ to $s_{m-1,t}$, but no more than by the measure of projects at stage m. In other words, by investing $i_{m,t}$, one can move no more than $s_{m,t}$ projects to the next stage of capital formation. Since a measure of projects which have not been started at time t, i.e. are J stages before completion, is infinite, we set $s_{J,t} = +\infty$, for all $t = 0, \ldots$

Let $\mathbf{s}_t = [s_{1,t}, \ldots, s_{J-1,t}] \in \times_m S_m \equiv \mathbf{S}$. Due to the above description, it is natural to state the following assumption.

Assumption 4 *Let* $\mathbf{S} \equiv \mathbf{R}_+^{J-1}$.

In the final stage of an investment fully operational capital is formed according to a standard law of motion:

$$k_{t+1} = (1 - \delta)k_t + \min\{s_{1,t}; \lambda_1 i_{1,t}\}, \tag{2}$$

where $\delta \in (0,1]$ denotes the rate of capital depreciation. Notably, the analyzed model is a generalization of an optimal growth model as well as of a standard time-to-build model by Kydland and Prescott [1982]. By setting $\lambda_1 = 1$ we obtain the former, while with $\lambda_J = 1$ we obtain the latter.

Having characterized the model we begin the analysis of a time-to-build economy with multi-stage investments. First we present the solution from the point of view of a social planner.

Social planner problem

We begin our analysis by defining and solving the centralized problem. We give sufficient conditions for the existence of an optimal solution in the economy, and characterize the main properties of the optimal allocations.

For any given initial level of capital $k_0 \in K$, any vector of measures of unfinished projects $\mathbf{s}_0 \in \mathbf{S}$, and any initial shock value $z_0 \in Z$, the social planner solves the following discounted infinite horizon problem:

$$v(k_0, \mathbf{s}_0, z_0) = \max_{\{c_t, n_t, l_t, \mathbf{i}_t\}_{t=0}^{+\infty}} E_0 \left(\sum_{t=0}^{+\infty} \beta^t u(c_t, n_t) \right), \tag{3}$$

s.t. $\forall t = 1, 2, \ldots$

$$f(k_t, l_t, z_t) \geq c_t + \sum_{m \in \mathcal{J}} i_{m,t}, \tag{4}$$

$$n_t + l_t = 1, \tag{5}$$

$$c_t \in X, n_t, l_t \in N, k_t \in K, \mathbf{s}^t \in \mathbf{S}, z_t \in Z. \tag{6}$$

$$s_{J,t} = +\infty, \tag{7}$$

$$k_{t+1} = (1 - \delta)k_t + \min\{s_{1,t}; \lambda_1 i_{1,t}\}, \tag{8}$$

$\forall m \in \mathcal{J}/\{J\},$

$$s_{m,t+1} = \max\{s_{m,t} - \lambda_m i_{m,t}; 0\} + \min\{s_{m+1,t}; \lambda_{m+1} i_{m+1,t}\}, \tag{9}$$

$$Prob(z_{t+1} \leq z' \,|\, z_t = z) = Q(z' \,|\, z), \tag{10}$$

where conditions (4)-(6) define a feasible action set for each *t*, and (7)-(10) characterize the law of motion of state variables. $\beta \in [0,1)$ denotes a discount factor.

For the sake of notation, later in the paper we will omit time subscripts where possible and denote a vector of state variables in a subsequent period by $(k', \mathbf{s}', z')$ respectively. Moreover, we reduce the number of decision variables by substituting decision variable n_t by l_t, such that $l_t = 1 - n_t$. In addition, since *u* is increasing, constraint (4) will be satisfied with equality, i.e. $c_t = f(k_t, l_t, z_t) - \sum_{m \in \mathcal{J}} i_{m,t}$, where $f(k_t, l_t, z_t) \geq \sum_m i_{m,t}$. Bellman's equation corresponding to problem (3), for any given given initial vector of state variables $k \in K, \mathbf{s} \in \mathbf{S}, z \in Z$, is

$$v(k, \mathbf{s}, z) = \max_{l,i}\Big(u\Big(f(k, l, z) - \sum_{m \in \mathcal{J}} i_m, 1 - l\Big) + \beta \int_z v(k', \mathbf{s}', z')Q(dz' \,|\, z)\Big), \tag{11}$$

$$s.t. \ f(k, l, z) - \sum_{m \in \mathcal{J}} i_m \geq 0, \tag{12}$$

$$l \in [0,1], \tag{13}$$

$$k' = (1 - \delta)k + \min\{s_1; \lambda_1 i_1\}, \tag{14}$$

$\forall m \in \mathcal{J}/\{J\},$

$$s'_m = \max\{s_m - \lambda_m i_m; 0\} + \min\{s_{m+1}; \lambda_{m+1} i_{m+1}\}. \tag{15}$$

Let $\Upsilon(k, \mathbf{s}, z) \subset [0,1] \times \mathbf{R}_+^J$ denote a set of all pairs $(l, \mathbf{i})$, satisfying conditions (12) and (13). Obviously, Υ is a continuous correspondence, with a compact and convex graph. We will refer to $\Upsilon(k, \mathbf{s}, z)$ as a feasible action set.

Equation (11) is standard to dynamic economic theory. Nevertheless, basing on the standard arguments, it is not obvious whether it has a solution. Note that as K and $\mathbf{S}$ are unbounded (both sets are equivalent to $\mathbf{R}_+$ and $\mathbf{R}_+^{J-1}$) respectively, it is not obvious whether there exists a solution to problem (11) in a set of continuous functions on $K \times \mathbf{S}$. Since the space of unbounded continuous functions is not bounded in the sup norm, Bellman operator $T: C(K \times \mathbf{S} \times Z) \to C(K \times \mathbf{S} \times Z)$,

$$
(Tv^n)(k, \mathbf{s}, z) = \max_{(l,i) \in \Upsilon(k,\mathbf{s},z)}
$$
$$
\left(u\Big(f(k, l, z) - \sum_{m \in \mathcal{J}} i_m, 1 - l \Big) + \beta \int_Z v^n(k', \mathbf{s}', z') Q(dz' \mid z) \right), \tag{16}
$$

where k' and $\mathbf{s}'$ are defined as in ( ) and ( ) respectively, might not converge in the space of bounded and continuous functions as $n \to \infty$, since Blackwell's *sufficient conditions for contraction* (see [Blackwell, 1965]) are not satisfied. Even though v^n could be somehow bounded on K using methods described among others in Stockey, Lucas and Prescott [1989], as it was mentioned in the introduction, there is no natural and economically admissible method to bound the value function on $\mathbf{S}$. For this reason, we refer to the work by Rincón-Zapatero and Palmero [2003, 2007, 2009], and Matkowski and Nowak [2009], where a method was developed for finding a unique solution to the Bellman equation in a space of unbounded continuous functions.

To be precise, Matkowski and Nowak [2009] prove the existence of a unique solution to the Bellman equation in a space of functions $\phi: X \to \mathbf{R}$, defined on some arbitrary set X, which are bounded in the following form:

$$
\| \phi \| := \sum_{j=1}^{+\infty} \frac{\| \phi \|_j}{m_j c^j},
$$

where $c^j > 1$, $m = \{m_j\}$ is an increasing, unbounded sequence of positive real numbers, and $\| \phi \|_j = \sup_{x \in K_j} |\phi(x)|$, where $\{K_j\}$ is a strictly increasing (in the sense of set inclusion) sequence of subsets of X, such that $X = \bigcup_{j=1}^{+\infty} K_j$. For further discussion, please refer to the original paper. Since our analysis will be restricted to continuous functions, we denote the space by $C_m(X)$.

Now we can present our first result.

Proposition 2.1 (Unique value function) *Let assumptions 1-4 be satisfied. There exists an increasing, unbounded sequence $m = \{m_j\}$ and a unique function $v^* \in C_m(K \times \mathbf{S} \times Z)$, which satisfies equation (11).*

Proof: Let $\{K_j \times \mathbf{S}_j\}$ be a strictly increasing (in the sense of set inclusion) sequence of compact subsets of $K \times \mathbf{S}$, such that $K \times \mathbf{S} = \bigcup_{j=1}^{+\infty} K_j \times \mathbf{S}_j$. Define $C_{mb}(K \times \mathbf{S} \times Z) := \left\{ \phi \in C(K \times \mathbf{S} \times Z) : \|\phi\|_j \le m_j, \forall j = 1, \dots \right\}$. Clearly $C_{mb}(K \times \mathbf{S} \times Z)$ is a closed subset of $C_m(K \times \mathbf{S} \times Z)$. Define T on $C_{mb}(K \times \mathbf{S} \times Z)$ as in (16). Due to the continuity assumptions imposed on u and f, compactness of $\Upsilon(\cdot)$, and the fact that f is bounded on Z, Tv is well defined. By the maximum theorem of Berge [1963], Tv is continuous on every set $K_j \times \mathbf{S}_j \times Z$. Moreover, it is continuous on $K \times \mathbf{S} \times Z$ (see remark 1(a) in Matkowski and Nowak [2009]). Thus T is mapping $C_{mb}(K \times \mathbf{S} \times Z)$ into itself. The rest follows Proposition 3 in Matkowski and Nowak [2009]. Q.E.D.

The uniqueness and continuity of a value function satisfying equation (11) is a powerful result, taking into an account that we are analyzing unbounded functions. Nevertheless, in order to make the analysis more resourceful, we would like to examine other properties of the value function, in particular its smoothness.

Our next result will concern the differentiability of the value function satisfying equation (11). Since the law on motion imposed on $\mathbf{s} \in \mathbf{S}$ includes maximum and minimum functions, which are not continuously differentiable at every point in its domain, it is not obvious whether $v^* \in \mathcal{C}^1(K \times \mathbf{S})$, $\forall z \in Z$. However, it is possible to reformulate the problem in order to obtain additional results.

First of all, observe that it is never optimal to set investment $i_m > s_m/\lambda_m$. To show this, assume the opposite. Let $\mathbf{i}' = \left(i_1, \dots, i_j', i_J\right)$, $i_{j'} > s_j/\lambda_j$, and $\mathbf{i} = (i_1, \dots, i_j, \dots, i_J)$, $(l, \mathbf{i})$, $(l, \mathbf{i}') \in \Upsilon(k, \mathbf{s}, z)$, for any k, $\mathbf{s}$ and z. Then,

$$u\left(f(k, l, z) - \sum_{m \ne j} i_m - i_{j'}, 1 - l\right) + \beta \int_z v(k', \mathbf{s}', z')Q\left(dz' \mid z\right)$$
$$< u\left(f(k, l, z) - \sum_{m \ne j} i_m - s_j/\lambda_j, 1 - l\right) + \beta \int_z v(k', \mathbf{s}', z')Q\left(dz' \mid z\right)$$
$$\le v(k, \mathbf{s}, z),$$

where k' and $\mathbf{s}'$ are defined as in (14) and (15) respectively. The first inequality is implied by the law of motion of $\mathbf{s}$ and monotonicity of u, and the second by the definition of v. The above observation makes it possible to reduce the problem to a much simpler case where all the constraints are linear and differentiable.

Denote a set of all $(l, \mathbf{i}) \in \Upsilon(k, \mathbf{s}, z)$, such that $\forall m \in \mathcal{J}$, $i_m \in [0, s_m/\lambda_m]$ by $\Gamma(k, \mathbf{s}, z)$. The problem in (11) can be reformulated as follows:

$$v(k, \mathbf{s}, z) = \max_{(l,i) \in \Gamma(k,s,z)} \left(u\left(f(k, l, z) - \sum_{m \in J} i_m, 1 - l\right) + \beta \int_z v(k', s', z)Q\left(dz' \mid z\right) \right), \quad (17)$$

$$where \ k' = (1 - \delta)k + \lambda_1 i_1, \quad (18)$$

$$\forall m \in \mathcal{J}/\{J\},$$
$$s'_m = s_m - \lambda_m i_m + \lambda_{m+1} i_{m+1}. \tag{19}$$

Obviously, by the aforementioned argument, (11) and (17) are equivalent. Let $h \colon K \times \mathbf{S} \times Z \to [0,1] \times \mathbf{R}^J$ denote the solution to (17), such that $h(k, \mathbf{s}, z) = (l^*, \mathbf{i}^*)$.

In order to prove our next result, we will change the decision variable in the above problem, so that each period consumer will determine the optimal levels of k', $\mathbf{s}$ and l instead of $(l, \mathbf{i})$. Using (18) and (19), it is straightforward to show that $\forall m, i_m = \frac{1}{\lambda_m}\left[(k'-(1-\delta)k) + \sum_{i=1}^{m-1}(s_{i'} - s_i)\right]$. (17) is then equivalent to

$$v(k, \mathbf{s}, z) = \max_{l,k',\mathbf{s}'}(u(f(k, l, z) - (k' - (1-\delta)k) -$$
$$\sum_{m=1}^{J}\frac{1}{\lambda_m}\sum_{i=1}^{m-1}(s_{i'} - s_i), 1 - l) \tag{20}$$
$$+ \beta \int_z v(k', \mathbf{s}', z')Q(dz' \,|\, z)),$$

$$s.t. \; f(k, l, z) - (k' - (1-\delta)k) - \sum_{m=1}^{J}\frac{1}{\lambda_m}\sum_{i=1}^{m-1}(s_{i'} - s_i) \geq 0, \tag{21}$$

$$(k' - (1-\delta)k) \in [0, s_1],$$
$$\forall m \in \mathcal{J}/\{J\}, \tag{22}$$

$$(k' - (1-\delta)k) + \sum_{i=1}^{m-1}(s_{i'} - s_i) \in [0, s_m]. \tag{23}$$

Let $\Theta(k, \mathbf{s}, z)$ be the set of all feasible $(l, k', \mathbf{s}')$ satisfying conditions (21)-(23). Moreover, let $g \colon K \times \mathbf{S} \times Z \to [0,1] \times K \times \mathbf{S}$ denote the solution to (20), such that $g(k, \mathbf{s}, z) = (l^*, k'^*, \mathbf{s}'^*)$.

The above reformulation of the problem makes it possible to examine additional properties of the value function and optimal policies. Moreover, it helps rule out the non-linear constraints, and reduce the problem to a standard formulation as in Stockey, Lucas and Prescott [1989]. We proceed to our next result.

Proposition 2.2 *Let Assumptions 1-4 be satisfied. Then v^* is increasing, and strictly concave on $K \times \mathbf{S}$. Moreover, for any $(k, \mathbf{s}, z)$, $g(k, \mathbf{s}, z)$ is unique, and $g(\cdot,\cdot, z)$ is a continuous function.*

Proof: We examine problem (20), and follow argumentation as in Stockey, Lucas and Prescott [1989] (see Theorems 4.7 and 4.8). Q.E.D.

The last result has strong implications. First of all, since v^* is increasing on $K \times \mathbf{S}$, this implies that the higher the level of capital and unfinished projects, the higher the discounted lifetime utility of the agent. The result is quite obvious in the case of capital; a higher level of capital increases production, and thus consumption, however it seems to be less intuitive when talking about the measures of unfinished projects. Even though the measure of capital in formation does not directly affect the level of output, it increases the lifetime utility through higher consumption. Observe that the higher the $\mathbf{s}$, the less resources have to be invested in order to complete the same level of operational capital. Since the output is divided solely to consumption and investments, the lower the total level of investments, the higher the consumption.

Even though the decision variables in problems (17) and (20) are different, the conclusion of proposition 2.2 holds for an argument maximizing problem (17). We proceed with the following remark.

Remark 2.1 *Let Assumptions 1-4 be satisfied. Then for any* $(k, \mathbf{s}, z)$, $h(k, \mathbf{s}, z)$ *is unique, and* $h(\cdot, \cdot, z)$ *is a continuous function.*

Proof: Due to Proposition 2.2, g is unique and continuous. Since $\forall m \in \mathcal{J}$, $i_m = \frac{1}{\lambda_m}\left[(k' - (1 - \delta)k) + \sum_{i=1}^{m-1}(s', - s_i)\right]$, $\mathbf{i}$ is unequivocally determined by k, k', $\mathbf{s}$, $\mathbf{s}'$. h is thus unique. Moreover, due to Berge's Maximum Theorem, $h(\cdot, \cdot, z)$ is continuous. Q.E.D.

The concavity of the value function is an important property, since it guarantees uniqueness of the optimal policy function. However, even though v^* is concave, it is not necessarily differentiable. Observe that the Inada conditions imposed on the utility function guarantee that optimal l is in the interior of the feasible policy set, as well as $f(k, l^*, z) > \sum_{m=1}^{L} i_m^*$. However, it gives no additional information about the optimal values of investments $\mathbf{i}$ and the future stock of incomplete projects $\mathbf{s}'$. It is plausible that either for some $m \in \mathcal{J}$ $i_m = s_m$, $i_m = 0$, or condition (22) is satisfied. In this case, the argument developed by Benveniste and Scheinkman [1979] would be invalid as the optimal policy would not be in the interior of the constraint set.

The lack of differentiability implies two substantial drawbacks. First, without a proper level of smoothness of v^*, one cannot study the optimal solution through the first-order conditions imposed on the gradient of v^*, not mentioning the inverse function theorem. Of course, since v^* is concave, one could always determine the solution using the first-order conditions imposed on the subgradients the value function, but since the subgradients of the non-smooth functions are not singletons, this would essentially complicate the computations.

Second, when decentralizing the optimal allocations, the first-order derivatives of the value function play a crucial role in determining prices in the economy, as it will be shown in section 2.3.

Fortunately, as presented in Rincón-Zapatero and Santos [2009], the question of differentiability can be determined even when the optimal policies are not in the interior of the constraint set. The authors develop sufficient conditions for the differentiability of the value function, without interiority assumptions imposed on argmax correspondences. In the following proposition, we use their result to prove the differentiability of the value function in a time-to-build economy with multi-stage investments.

Proposition 2.3 $v^*(\cdot, \cdot, z) \in \mathcal{C}^1\ (K \times S)$ *for all* $z \in Z$.

The proof of the above proposition is conducted in a similar way to the proof of theorem 3.1, and corollary 3.1 in Rincón-Zapatero and Santos [2009]. The argument is discussed in detail in the appendix.

The above result makes it possible to find a solution to the optimization problem via the first-order conditions of the corresponding Lagrangian. We state the following remark.

Remark 2.2 *Let* (l^*, k'^*, s'^*) *be an optimal solution to (20), and* $\mu_k^*, \phi_k^*, \left\{\mu_m^*\right\}_{m \in \mathcal{J}/J}$, $\left\{\phi_m^*\right\}_{m \in \mathcal{J}/J}$ *the Lagrange multipliers corresponding to constraints (22), and (23). Then*

$$\beta \int_z v_k^*(k'^*, s'^*, z')Q(dz' \mid z) + \left(\mu_k^* - \phi_k^*\right) + \sum_{m=1}^{J-1}\left(\mu_m^* - \phi_m^*\right) = u_c(c^*, n^*),$$

$$u_c(c^*, n^*)f_l(k, l^*, z) = u_n(c^*, n^*),$$

$$(\forall m), \beta \int_z v_{s_m}^*(k'^*, \mathbf{s}'^*, z')Q(dz' \mid z) + \sum_{i=1}^{m-1}(i-1)\left(\mu_i^* - \phi_i^*\right)$$

$$= u_c(c^*, n^*)\left[\sum_{m-1}^{J-1}\frac{1}{\lambda_m}(m-1)\right].$$

Moreover, by the envelope theorem

$$v_k^*(k, \mathbf{s}, z) = u_c(c^*, n^*)\left[f_k(k, l^*, z) - (1 - \delta)\right]$$

$$-(1 - \delta)\left[\mu_k^* + \sum_{m=1}^{J-1}\mu_m^*\right] + (1 - \delta)\left[\phi_k^* + \sum_{m=1}^{J-1}\phi_m^*\right]$$

and for all $m = 1, ..., J - 1$

$$v_{s_m}^*(k, \mathbf{s}, z) = u_c(c^*, n^*) - \sum_{m=1}^{J-1}\mu_m^*(m-1) + \sum_{m=1}^{J-1}\phi_m^* m,$$

where

$$c^* = f(k, l^*, z) - \left(k'^* - (1 - \delta)k\right) - \sum_{m=1}^{J} \frac{1}{\lambda_m} \sum_{i=1}^{m-1} \left(s'^*_i - s_i\right),$$

$$\text{and } l^* = 1 - n^*.$$

Proof: We use a straightforward application of the Karush-Khun-Tucker theorem. Q.E.D

Even though v^* was proven to be differentiable, analyzing the first-order derivatives of the problems in (17) and (20) might not be a straightforward method for solving the optimization problems. Since the solution to (17), as well as to (20) might be at the bound of the feasible action set $\Gamma(k, \mathbf{s}, z)$ and $\Theta(k, \mathbf{s}, z)$ for some $(k, \mathbf{s}, z)$ respectively, the optimal Lagrangian multipliers might be hard to determine, and so the properties of the policy function may be hard to analyze using the first-order conditions.

With the lack of standard methods for determining the monotonicity of optimal policies, one could always refer to more general tools, like the ones offered by lattice programming. Unfortunately, it is commonly known that for any k, $\mathbf{s}$, z, $\Gamma(k, \mathbf{s}, z)$ and $\Theta(k, \mathbf{s}, z)$, incorporated with a standard order on $\mathbf{R}^n$ are not lattices, which means that the standard monotone comparative statics methods developed by [48, 38, 31, 46, 47, 29] and others, are also improper for our analysis. For this reason, we will refer to a work by Quah [2007], which deals with the comparative statics of optimization problems, where a feasible action set might not be a lattice in the standard sense. In particular, Quah [2007] presents a new ordering suitable to simplexes, which under some concavity conditions of the payoff function preserves the monotonicity properties of the argmax correspondences. We proceed with our next result.

Proposition 2.4 *Let assumptions 1-4 be satisfied. Then $\forall z \in Z$, g is non-decreasing on $\mathbf{S}$ in the natural ordering on $\mathbf{R}^n$*[5].

The proof is discussed in the appendix. The last proposition characterizes the monotonicity of optimal solutions with respect to the measure of unfinished projects. It implies that the higher the measure of projects, the higher are the decisions of consumers concerning not only the measure of future projects, but also the time spent on labor.

However, the monotonicity takes place only with respect to $\mathbf{s}$. Since the payoff function is not necessarily quasi-supermodular in (l, k), there is no guarantee that the optimal level of labor is increasing with k, as both the income and substitution effects might take place when the level of capital increases,

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[5]   Take any $x, y \in \mathbf{R}^n$. We say that x is greater than y in the natural order (component-wise), and denote $x \geq y$, if $\forall i$, $x_i \geq y_i$.

thus the monotonicity of g on k takes place only in some special cases of utility functions. We present an example of such preferences in remark 2.3.

On the other hand, the monotonicity of h is ambiguous even when g is non-decreasing. This means that the optimal level of investments does not have to be increasing in $\mathbf{s}$ even when the optimal level of $\mathbf{s}'$ is. Again due to the substitution and income effects between different stages of investment, the direction of changes in optimal investments in unequivocal.

In some special cases, the optimal policy function can be increasing on $K \times \mathbf{S}$. An example of this is presented below. In general, the result is valid whenever one can prove the monotonicity of the optimal level of labor with respect to capital. We present the following assumption.

Assumption 5 (Inelastic labor) *Let u be defined as in assumption. $u_n \equiv 0$.*

Observe that once the utility of consumer is constant with respect to leisure, the optimal level of labor is always equal to 1. This in particular implies that the optimal level of labor is non-decreasing in k. This enables us to state the following remark.

Remark 2.3 *Let assumptions 1-5 be satisfied. Then $\forall z \in Z$, (i) g is non-decreasing on $K \times \mathbf{S}$ in the natural ordering on $\mathbf{R}^n$.*

If f is concave and increasing on Z, and $Q(dz' | \cdot)$ is concave on Z, then (ii) g are is non-decreasing in $K \times \mathbf{S} \times Z$ in the natural ordering on $\mathbf{R}^n$.

Proof: The proof is similar to the proof of proposition 2.4. We begin with (i). This time observe that (20) is supermodular on $((l, k', \mathbf{s}'), (k, \mathbf{s}))$, since (20) has constant differences on $(l, (k, \mathbf{s}))$. This implies that (20) is *C-supermodular*.

Since $\Theta(k, \mathbf{s}, z)$ is ascending in the *C-flexible set order*, as shown in the proof of proposition 2.4, $g(\cdot, \cdot, z)$ is non-decreasing, by Corollary 1 in Quah [2007].

To prove (ii) we use the same argument, which is valid due to the concavity and monotonicity of $f(k, l, \cdot)$ and the concavity of $Q(dz' | \cdot)$. Q.E.D.

We conclude this section by a weak result concerning the existence of a steady state in the analyzed economy.

Remark 2.4 (Trivial steady state) *For any constant shock level $z \in Z$, let a steady state be defined by a triple $(l^{ss}, k^{ss}, \mathbf{s}^{ss})$, such that $g(k^{ss}, \mathbf{s}^{ss}) = (l^{ss}, k^{ss}, \mathbf{s}^{ss})$.*

There exists a trivial steady state $(l^{ss}, k^{ss}, \mathbf{s}^{ss}) = \theta$, where θ is the zero vector.

We omit the proof.

Decentralized economy

In this section, we characterize an example of an economy that decentralizes the optimal allocations characterized in the previous part.

The economy consists of three sectors: consumers, the production sector, and investment sector. We characterize each one of them respectively.

Consumers

The problem of a representative consumer is to maximize his lifetime utility, with respect to every period consumption and leisure. The preferences are defined as in assumption 1. Each period t, and in each state z, the consumer is in possession of a unit of time and a portfolio of investment assets, denoted $s_t \in \mathbf{R}_+^J$. Let $\mathbf{a}_t = (k_t, a_1, ..., a_{J-1})$, where we will refer to k_t as to a capital asset.

At the beginning of each period, the consumer receives a return from his asset portfolio $\mathbf{a}_t$, at rates given by vector $\mathbf{q}_t \in \mathbf{R}_+$. Next, he lends capital k_t to the production sector at price r_t and receives remuneration from labor w_t. The revenue is spent on a single consumption good c_t, with the price normalized to 1, and a new portfolio $\mathbf{a}_{t+1}$ at price $\mathbf{p}_t$, which will be due in the subsequent period. For a given portfolio $\mathbf{a}_t$, given state z_0, and any infinite sequence of prices $\{\mathbf{p}_t, \mathbf{r}_t, w_t\}_{t=0}^{+\infty}$, the optimization problem of the consumer is,

$$v^{CE}(\mathbf{a}_0, z_0) = \max_{\{c_t, n_t, l_t, \mathbf{a}_t\}_{t=0}^{+\infty}} E_0\left(\sum_{t=0}^{+\infty} \beta^t u(c_t, n_t)\right), \tag{24}$$

$$s.t. \sum_{t=0}^{\infty}\left[\mathbf{q}_t \cdot \mathbf{a}_t + r_t k_t + w_t l_t\right] \geq \sum_{t=0}^{\infty}\left[c_t + \mathbf{p}_t \cdot \mathbf{a}_{t+1}\right], \tag{25}$$

$$\forall t = 1, 2, ...; \forall z_t \in Z,$$
$$n_t + l_t = 1, \tag{26}$$
$$c_t \in X, n_t, l_t \in N, \mathbf{a}_t \in K \times \mathbf{S}, z_t \in Z.$$

where operator E_0 is defined by transition kernel Q, as in assumption 3. As in the previous section, the existence and characterization of the equilibrium allocations and prices will be derived using the dynamic programming arguments. For this reason, the Bellman equation corresponding to the above problem, for any given $\mathbf{a}$, z is as follows. To simplify the notation, we omit time subscripts, and denote the future state variables by $\mathbf{a}'$, z'.

$$v^{CE}(\mathbf{a}, z) = \max_{\{c, n, \mathbf{a}'\}}\left(u(c, n) + \beta \int_z v(\mathbf{a}', z')Q(dz' \mid z)\right), \tag{27}$$

$$s.t. \ \mathbf{q} \cdot \mathbf{a} + rk + wl \geq c + \mathbf{p} \cdot \mathbf{a}', \tag{28}$$

$$n + l = 1,$$
$$c \in X, n, l \in N, \mathbf{a}, \mathbf{a}' \in K \times \mathbf{S}, z_t \in Z. \tag{29}$$

Corollary 2.1 *Let assumption 1 be satisfied, then for any a_t, and any vector of prices $\{p_t, q_t, r_t, w_t\}_{t=0}^{+\infty}$,*

(i) there is a unique solution v^{CE} to (27). Moreover, v^{CE} is continuously differentiable, strictly increasing and strictly concave;

(ii) let $g: K \times S \to [0,1] \times K \times S$ be an argument solving problem (27). g is a continuous function, non-decreasing on S.

Proof: To prove both (i) and (ii) we use the same arguments as in the proofs of proposition 2.1, 2.2, and 2.4. Q.E.D.

Production sector

The production sector in our economy is presented in a very standard way. We assume that an infinite mass of identical agents produce a unique consumption good c_t, using technology f, described in assumption 2. Taking the prices of the consumption good and input factors as given, each firm buys factors of production and determines production plans in order to maximize its profit, given the current productivity shock z_t.

With the price of the consumption good normalized to 1, and the prices of inputs w_t and r_t, each firm solves

$$\max_{K_t, L_t} f(K_t, L_t, z_t) - w_t L_t - r_t K_t. \tag{30}$$

Since the technology of production exhibits a constant return to scale (see assumption 2), in an equilibrium profits will be drawn to zero.

Investment sector

The investment sector consists of an infinite number of identical firms. Each firm is investing in current unfinished projects. Taking the prices as given, the firms are buying assets s_t at a given price q_t, which consumers bought the previous period, and invest them in different stages of unfinished projects of capital accumulation. The investment profile will be denoted by $i_t = (i_{1,t}, ..., i_{J,t})$, where $i_{j,t}$ denotes an investment in a project j periods before completion. The technology of investment is described by

$$s_{m,t+1} = s_{m,t} - \lambda_m i_{m,t} + \lambda_{m+1} i_{m+1,t},$$

and

$$k_{t+1} = (1 - \delta)k_t + \lambda_1 i_{1,t},$$

where $s_{m,t+1}$ and k_{t+1} denotes the measure of new projects and capital, after the current period investment took place.

By the end of each period, firms sell a new portfolio of assets to consumers at price $\mathbf{p}_t$, in order to maximize profits over $\mathbf{s}_{t+1}$, $\mathbf{s}_t$ and $\mathbf{i}_t$

$$\mathbf{p}_t \cdot \mathbf{s}_{t+1} - \mathbf{q}_t \cdot \mathbf{s}_t - \sum_{k=1}^{J} i_k,$$

subject to $s_{m,t+1} = s_{m,t} - \lambda_m i_{m,t} + \lambda_{m+1} i_{m+1,t}$, and $k_{t+1} = (1-\delta)k_t + \lambda_1 i_{1,t}$.

As in the case of the production sector, it is easy to verify that the investment technology exhibits constant returns to scale, and so in an equilibrium, the profit of all investment firms will also be equal to zero.

General equilibrium

Once we have defined an economy of our interest, we present the definition of an equilibrium.

Definition 2.1 (Equilibrium) *For any sequence* $\{z_t\} \in Z^\infty$, *an equilibrium is a tuple of allocations and prices* $\left\{r_t^*, \boldsymbol{q}_t^*, \boldsymbol{p}_t^*, w_t^*, c_t^*, n_t^*, l_t^*, \boldsymbol{a}_t^*, K_t^*, L_t^*, \boldsymbol{s}_t^*, \boldsymbol{i}_t^*\right\}_{t=0}^{\infty}$, *such that*
(i) Consumers optimize their lifetime utility, taking prices as given,

$$\left\{c_t^*, n_t^* \, l_t^*, \mathbf{a}_t^*\right\}_{t=0}^{+\infty} \in \arg \max_{\{c_t, n_t, l_t, \mathbf{a}_{t+1}\}_{t=0}^{+\infty}} E_0\left(\sum_{t=0}^{+\infty} \beta^t u(c_t, n_t)\right), \qquad (31)$$

$$\forall t = 1, 2, \ldots,$$
$$n_t + l_t = 1,$$
$$c_t \in X, n_t, l_t \in N, \boldsymbol{a}_t \in K \times S.$$

(ii) Production firms optimize their profits, $\forall t$,

$$\left(K_t^*, L_t^*\right) \in \arg \max_{K_t, L_t} f(K_t, L_t, z_t) - w_t^* L_t - r_{k,t}^* K_t. \qquad (32)$$

(iii) Investment firms optimize their profits, $\forall t$,

$$\left(\boldsymbol{s}_{t+1}^*, \boldsymbol{s}_t^*, \boldsymbol{i}_t^*\right) \in \arg \max_{\boldsymbol{s}_{t+1}, \boldsymbol{s}_t, \boldsymbol{i}_t} \boldsymbol{p}_t^* \cdot \boldsymbol{s}_{t+1} - \boldsymbol{q}_t^* \cdot \boldsymbol{s}_t - \sum_{k=1}^{J} i_k, \qquad (33)$$

$$s.t. \ \forall m = 1, \ldots, J,$$
$$s_{m,t+1} = s_{m,t} - \lambda_m i_{m,t} + \lambda_{m+1} i_{m+1,t},$$
$$k_{t+1} = (1-\delta)k_t + \lambda_1 i_{1,t}.$$

(iv) Markets clear, $\forall t$,

$$f\big(K_t^*, L_t^*, z_t\big) = c_t^* + \boldsymbol{p}_t^* \cdot \boldsymbol{a}_{t+1}^*, \; \boldsymbol{a}_t^* = \boldsymbol{s}_t^*, \; K_t^* = k_t^*, \; L_t^* = l_t^*, \; n_t^* + l_t^* = 1.$$

The above definition of a general equilibrium is standard and refers directly to the notion of equilibrium presented by Arrow and Debreu in the infinite dimensional setting. Condition (i) requires that consumers maximize their lifetime utility with respect to the budget constraints. The second and third conditions require that firms maximize their profits, while (iv) imposed the market clearing condition.

Once we have defined the notion of an equilibrium, we proceed to the next result, stating the existence of an equilibrium.

Theorem 2.1 *Let assumptions 1-4 be satisfied. Then, the equilibrium defined in definition 2.1 exists.*

Proof: The presented economy is a standard, convex, infinite horizon economy with a space of goods defined by L_∞. It is sufficient to show that the presented economy is only a special case of the economy presented in Bewley [1972]. We refer the reader to the original paper. Q.E.D.

Eventually, we present a result showing that the allocations obtained in the decentralized economy are equivalent to the optimal allocation chosen by the social planner. Moreover, we provide a constructive result that provides a method for computing equilibrium prices. The idea is based on a work by Negishi [1960], who was the first to show that general equilibrium allocations can be obtained through a maximization problem of a social planner, where prices were determined by the Lagrange multipliers of the corresponding problem. The result was extended to economies with an infinite horizon by Papageorgiou [1960], who obtained the same result using the dynamic programming approach. We present the following corollary based on their results.

Corollary 2.2 *Let assumptions 1-4 be satisfied. Then the first and second welfare theorems hold. Moreover,* $\big(r_t^* + q_{k,t}^*, \boldsymbol{q}_{-k,t}^*\big) = \nabla_a v^{CE}\big(\boldsymbol{a}_t^*, z_t\big),$ *where* $q_{k,t}^*$ *and* $\boldsymbol{q}_{-k,t}^*$ *denote the first element and a vector of all but the first element of* $\boldsymbol{q}_t^*$ *respectively.*

Proof: First we proceed with a proof of the first welfare theorem. The solution of the production sector problem leads to $r_t^* = f_K\big(K_t^*, L_t^*\big)$, and $w_t^* = f_L\big(K_t^*, L_t^*\big)$, for all t. Moreover, due to constant returns to scale of the production technology, $(\forall t) \, f\big(K_t^*, L_t^*\big) = r_t^* K_t^* + w_t^* L_t^*$. On the other hand, the maximization of the investment sector and constant returns to scale of the investment technology imply that $\mathbf{p}_t^* \cdot \mathbf{s}_{t+1}^* - \mathbf{q}_t^* \cdot \mathbf{s}_t^* = \sum_{k=1}^{J} i_{k,t}^*$.

Now we proceed to the consumer problem. Since the utility function is concave and increasing, due to the Karush-Khun-Tucker theorem, the consumer budget constraint is binding, i.e. $\sum_{t=0}^{\infty}\left[\mathbf{q}_t^* \cdot \mathbf{a}_t^* + r_t^* k_t^* + w_t^* l_t^*\right] = \sum_{t=0}^{\infty}\left[c_t^* \mathbf{p}_t^* \cdot \mathbf{a}_{t+1}\right]$. Moreover, due to production and investment sector optimization, as well as capital and labor market clearance, we conclude that $(\forall t)\, f\left(k_t^*, l_t^*, z_t\right) = \sum_{k=1}^{J} i_{k,t}^* + c_t^*$. Since $(\forall t)\mathbf{s}_t^* = \mathbf{a}_t^*$, and due to investment sector optimization, we conclude that unfinished capital stock allocations are equivalent to those determined by the social planner, which completes the proof.

The proof of the second welfare theorem is conducted analogously to the proof in Papageorgiou [1960], that is why we refer the reader to the original paper in order to safe space.

Eventually, since v^{CE} is differentiable, by the work on Negishi [1960] we conclude that the equilibrium prices are equal to the gradient of the consumer value function evaluated at the equilibrium allocation, i.e. $\left(r_t^* + q_k^*, \mathbf{q}_{-k,t}^*\right) = \nabla_a v^{CE}\left(\mathbf{a}_t^*, z_t\right)$, which completes the proof. Q.E.D.

Our final result presents two constructive conclusions. First, giving an example of an economy that decentralizes the optimal allocation makes it possible to analyze the equilibrium allocations through the optimization problem of a social planner. Moreover, since the consumer value functions are equivalent in the case of both a centralized and decentralized economy, the vector of the equilibrium prices is in fact equivalent to the gradient of the value function obtained through social planner optimization. In other words, finding a solution to the centralized problem is sufficient to characterize the allocations and prices in the decentralized economy.

Conclusion

In our paper we presented a constructive method for studying dynamic general equilibrium models of time-to-build economies with multiple-stage investments, thus generalizing the results obtained in the seminal paper by Kydland and Prescott [1982]. Using several results in dynamic programming theory and monotone operators (e.g. by Matkowski and Nowak [2009], Quah and Strulovici [2009] and Rincón-Zapatero and Santos [2009]), we construct a convergent algorithm for computing optimal allocations of the model, as well as propose an economy that decentralizes the allocations in the Arrow-Debreu general equilibrium. Eventually, we present comparative statics results, useful for a more general analysis of the problem, and discuss the efficiency of the presented economy.

Appendix

Proof of proposition 2.3: We replicate the proof of theorem 3.1, and corollary 3.1 in Rincón-Zapatero and Santos [2009]. We only have to verify whether all four conditions D1-D4 presented in the paper are satisfied.

Assumption D1 is satisfied by Assumptions 1 and 2.

For condition D2 to be satisfied, it is necessary to show that optimal policies are in the interior of $N \times K \times \mathbf{S}$. Due to the Inada conditions imposed on u, the optimal level of labor $l^* \in (0,1)$. Condition (22) implies that $k' > 0$, whenever $\delta < 1$. Otherwise $k' > 0$, by the Inada condition imposed on u, and the fact that $f(0, l, z) = 0$, all $l \in N$, and $z \in Z$. It is sufficient to show that for any m, $s'_m = 0$ is never optimal. Assume the contrary. For some $(k, \mathbf{s}, z)$, and some m, let k'_0, l_0 and $\mathbf{s}_0' = \left(s'_{1,0}, ..., s_{j-1,0}', 0, s_{j+1,0}, ..., s_{J,0}'\right)$ be a solution to (20). Then,

$$v(k, \mathbf{s}, z) = u\left(f(k, l, z) - (k' - (1 - \delta)k) - \sum_{m=1}^{J} \frac{1}{\lambda_m}\sum_{i=1}^{m-1}(s_{i,0}' - s_i), 1 - l_0 \right)$$

$$+ \beta \int_z v(k_{0'}, \mathbf{s}_{0'}, z')Q(dz' \mid z) =$$

$$u\left(f(k, l, z) - (k' - (1 - \delta)k) - \sum_{m=1}^{J} \frac{1}{\lambda_m}\sum_{i=1}^{m-1}(s_{i,0}' - s_i), 1 - l_0 \right) \quad (34)$$

$$+ \beta\left(u\left(f(k_{0'}, l'', z') - (k'' - (1 - \delta)k'_0) - \sum_{m=1}^{J} \frac{1}{\lambda_m}\sum_{i=1}^{m-1}(s_i'' - s'_{i,0}), 1 - l'' \right) \right.$$

$$\left. + \beta \int_z v(k'', s'', z'')Q(dz'' \mid z') \right)Q(dz' \mid z)$$

for some $(k'', \mathbf{s}'', z'')$ satisfying (20) in the subsequent period. Let $\mathbf{s}'' = (s''_1, ..., s_J'')$, and $\mathbf{s}_\varepsilon' = (s'_{1,0}, ..., s_{j-1,0}', \varepsilon, s_{j+1,0} - \varepsilon, ..., s_{J,0}')$, $s_{j+1,0} \geq \varepsilon$. Observe that any path of unfinished capital $\{\mathbf{s}_\varepsilon', \mathbf{s}''\}$ is feasible, and yields the same lifetime utility for the consumer as $\{\mathbf{s}', \mathbf{s}''\}$. This means that $\mathbf{s}_\varepsilon'$ must also be optimal, which, due to Proposition 2.2, yields a contradiction. We have shown that, for any $\mathbf{s}_0'$, we can construct a vector $\mathbf{s}_\varepsilon' \neq \mathbf{s}_0'$, which yields the same level of utility. This implies that $\mathbf{s}_0'$ is not optimal. Assumption D2 is satisfied.

Refer to constraints (21)-(23). Since they are linear with respect to the decision variables, and linearly independent, Assumption D3 is satisfied. Since the constraints are linear, their derivatives are constant, as a result of which the tansversality condition in Assumption D4 in the paper always holds with $B_0 = 0$. The proof is complete. Q.E.D.

Before stating the proof of proposition 2.4, we present the following two lemmas.

Lemma 2.1 *Let sets A and B be ascending in the C-flexible set order (see [Quah, 2007]). Then A ∩ B is also ascending in the C-flexible set order.*

The proof is trivial and is consequently omitted. The second lemma presents a class of sets that is always isotone in the *C-flexible order*.

Lemma 2.2 *Let sets* $A := \{x \in \mathbf{R}_+^n : a \leq \alpha^T x \leq b\}$, *where* α *is a vector in* $\mathbf{R}^n$, a, b *are positive scalars, and* $a \leq b$. *Then A is ascending in the C-flexible set order in a and b.*

Proof: Observe that A is defined by two hyperplanes. Let $H_a = \{x \in \mathbf{R}_+^n : a \geq \alpha^T x\}$ and $H_b = \{x \in \mathbf{R}_+^n : b \geq \alpha^T x\}$. Obviously H_a and H_b are ascending in the *C-flexible set order* in a and b respectively, as well as $H_{a'} := \mathbf{R}_+^n / H_a$. Moreover, $A = H_{a'} \cap H_b$. The rest follows Lemma 2.1. Q.E.D.

Now we proceed to the proof of Proposition 2.4.

Proof of proposition 2.4: Due to Proposition 2.2, v^* is concave. Moreover, it is easy to verify that (20) is supermodular in $(l, k', \mathbf{s}')$, and has increasing differences in $((l, k', \mathbf{s}'), (\mathbf{s}))$. This implies that (20) is *C-supermodular* (see Proposition 2 in [Quah, 2007]).

Refer to the constraints of problem (20). (21) is a standard budget constraint that is ascending in the *C-flexible set order* on $K \times \mathbf{S} \times Z$ (see [Quah, 2007]). Observe that each of the constraints (22) and (23) is defined by two parallel hyperplanes. Due to Lemma 2.2, they define a set ascending in the *C*-flexible set order on $(l, k, \mathbf{s})$. Since $\Theta(k, \mathbf{s}, z)$ is defined by an intersection of sets defined by constraints (21)-(23), by Lemma 2.2 it is also ascending in the *C-flexible set order*.

The rest follows Corollary 1 in Quah [2007]. Q.E.D.

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ON TIME-TO-BUILD ECONOMIES
WITH MULTIPLE-STAGE INVESTMENTS

S u m m a r y

The paper presents a constructive method for studying dynamic general equilibrium models with time-to-build technology. In the work, the author modifies a seminal work by Kydland and Prescott [1982] by allowing agents to make multi-stage investments. In this framework, consumers decide on how much to consume each period, and how much to invest in future capital, which will yield them income in future periods. Unlike in Kydland and Prescott, agents are allowed to postpone, stop or accelerate the investment by devoting a certain amount of their product to investments, which allows consumers to smooth investment expenses throughout the investment process rather than incur the whole cost at the beginning of the investment, as in Kydland and Prescott.

The analysis of the model poses several technical issues, which are tackled in the article. First, the premises of the model exhibit unbounded returns, which makes several

dynamic programming arguments invalid. Second, since the imposed assumptions do not exclude boundary solutions, the consumer value function of the underlying problem might not be differentiable, as the standard Benveniste and Scheinkman [1979] argument does not hold. Finally, as the feasible action set is not a lattice, standard monotone comparative statics cannot be introduced in order to bring about the monotonicity results concerning the policy functions in the model.

Using dynamic programming methods and monotone operators theory (by Matkowski and Nowak [2009], Quah and Strulovici [2009], and Rincón-Zapatero and Santos [2009]), the author presents a convergent algorithm for the computation of optimal allocations of the model, as well as proposes an economy that decentralizes the allocations in the Arrow-Debreu general equilibrium. Eventually, the author presents some comparative statics results useful for a more general analysis of the problem, and discusses the efficiency of the presented economy.

Keywords: time to build, dynamic general equilibrium, monotone comparative statics