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A Note on Bertrand Competition under Quadratic Cost Functions

Abstract: The authors focus on a model of competition known in economics as Bertrand competition. They investigate how the outcome of Bertrand competition changes when linear cost functions are replaced by quadratic functions in the model.

Named after French mathematician Joseph Louis Francois Bertrand (1822-1900), Bertrand competition describes interactions among firms and a market situation in which firms make their output and pricing decisions based on the assumption that their competitors will not change their own prices.

Prokop, Ramsza and Wiśnicki show that the introduction of quadratic cost functions in the model leads to a qualitative change in competition between firms. In this case, the standard assumption that a firm with a lower price is interested in taking over the entire market is not only unrealistic (even in the absence of capacity constraints), but also irrational from the profit-maximization viewpoint, the authors say. Therefore the results of previous studies are misleading, according to Prokop, Ramsza and Wiśnicki. They argue that the so-called Bertrand duopoly model should be adjusted to better capture the behavior of firms under quadratic cost functions. The authors relax the assumption that the output of firms is determined by existing demand. They offer a modified Bertrand duopoly model in which firms competing in prices are free to choose their level of production depending on the market demand for their products at specific prices. "Under the modified assumptions, the static price competition game of identical duopolists has no symmetric equilibrium in pure strategies," the authors conclude. Their research shows that, under quadratic cost functions, it is possible to expect price fluctuations on oligopolistic markets rather than a stable equilibrium situation described by the standard model of price competition.

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Introduction

It has been widely known that the static Bertrand competition in prices among identical firms in oligopoly leads to the outcome similar to perfectly competitive market. This result was based on the assumption of constant marginal costs of supply. Clearly, it would be interesting to see how robust this conclusion is to changes in the cost functions of firms.

The objective of this note is to show the key differences in the outcome of Bertrand competition under different cost functions of firms. In particular, we consider the homogenous duopoly competition when the cost functions are quadratic, and compare it with the results of the standard Bertrand models with demand-determined output discussed in the previous literature, summarized by, for example, Dastidar [1995], or Satoh and Tanaka [2013].

We show that the introduction of quadratic cost functions leads to a qualitative change in the competition between firms. In this case, the assumption that the firm with the lower price is interested in taking over the entire market is not only unrealistic (even in the absence of capacity constraints), but simply irrational from the profit-maximization viewpoint. Therefore the results of the analysis provided by the preceding literature are misleading. More specifically, we show that the Bertrand duopoly model should be adjusted to better capture the firms' behavior under the quadratic cost functions. We prove that the modified price competition game of duopolists characterized by identical quadratic cost functions has no symmetric equilibrium in pure strategies.

In the next section, we briefly review the standard Bertrand competition of firms with identical and constant marginal costs. The following section focuses on the model with quadratic cost functions of firms. In that section, we consider two different specifications of firms' behavior when their prices differ: the standard approach found in the existing literature, and the modified framework suggested in our paper. Conclusions are formulated in the last section.

The standard Bertrand duopoly

More than a hundred years ago, Bertrand [1883] pointed out that firms compete primarily in prices. In the simplest static model of price competition, two companies (duopoly) produce homogeneous good at identical and constant marginal costs c. Moreover, it is assumed that there are no capacity constraints, so that each firm would be able to satisfy the entire market demand.

The companies set prices of their products simultaneously and independently. Consumers buy the goods at the lowest quoted price. In the case of identical prices set by the suppliers, the demand is split equally between the two firms. Thus the demand for the products of firm *i* is given by:

$$D_i(p_i, p_j) = \begin{cases} D(p_i) & \text{for } p_i < p_j \\ \frac{1}{2}D(p_i) & \text{for } p_i = p_j \\ 0 & \text{for } p_i > p_j \end{cases} \tag{1}$$

Each firm maximizes its own one-period profit.

In the above game the Nash equilibrium is defined as a pair of strategies p_1^* and p_2^* such that the following two conditions are satisfied:

$$\pi_1(p_1^*, p_2^*) \geq \pi_1(p_1, p_2^*) \text{ for all } p_1, \tag{2}$$

$$\pi_2(p_1^*, p_2^*) \geq \pi_2(p_1^*, p_2) \text{ for all } p_2. \tag{3}$$

It means that, in the Nash equilibrium none of the firms has incentives to change the price of its product, given the price of its competitor.

Firm *i* would like to set its price slightly below the competitor's price. That would allow firm *i* to take over the entire market demand. Only when the rival's price is set at the level equal marginal costs, firm *i* will have no incentives to further undercut prices. Thus, it is easy to check that the only Nash equilibrium entails a pair of strategies p_1^* and p_2^* such that:

$$p_1^* = p_2^* = c. \tag{4}$$

In the above equilibrium, each firm earns zero profit. This result is called "Bertrand paradox"[1]: the price competition between just two firms is enough to generate the outcome of perfectly competitive market.

The Bertrand paradox disappears when we relax some of the assumptions of the basic model. There are three key directions of modification. First, the suppliers may have capacity constraints, which will not allow them to cover the entire market demand when they undercut prices. Second, by considering an infinite horizon for the competition between firms, the incentives for the firms to undercut in the short-run could be eliminated. Finally, when products offered by firms are differentiated, then even the sharp price competition may generate positive profits for both suppliers. In each of the above modifications, a symmetric Nash equilibrium in pure strategies exists and constitutes a good basis for the prediction of the firms' behavior in oligopoly.[2]

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1 It should not be confused with the Bertrand paradox known in the probability theory; see Bertrand [1889].

2 Compare, for example, Tirole [1988, pp. 211–212].

As mentioned at the beginning of this note, we will focus on the modification of the assumption about the cost functions of the suppliers. Instead of the linear form, we consider the quadratic cost function, i.e., the marginal costs will not be constant any more.[3]

The model with quadratic cost functions

The standard approach

Again consider an industry comprised of two firms: 1 and 2. Each of them produces an identical good at quantities q_1 and q_2, respectively. Market demand for the product is given as a linear function:

$$D(p) = a - p, \tag{5}$$

where p $(0 \le p \le a)$ denotes the market price, and a $(a > 0)$ is a given parameter reflecting the market size.

Each firm is characterized by the quadratic cost function

$$C(q_i) = q_i^2. \tag{6}$$

In the standard model of Bertrand competition it is assumed that the firms set prices simultaneously and independently, and the output is determined by the market demand. Therefore the profits are computed as follows

$$\pi_i(p_i, p_j) = \begin{cases} D(p_i)p_i - (D(p_i))^2 & \text{for } p_i < p_j \\ \frac{1}{2}D(p_i)p_i - \left(\frac{1}{2}D(p_i)\right)^2 & \text{for } p_i = p_j \\ 0 & \text{for } p_i > p_j \end{cases} \tag{7}$$

In the symmetric Nash equilibrium (if it exists) both firms will choose identical price level, denote it by p^*. Then, the profit of each firm will amount to

$$\pi_i^* = \frac{a - p^*}{2} p^* - \left(\frac{a - p^*}{2}\right)^2. \tag{8}$$

If one of the firms, say firm 1, reduced its price slightly below p^*, it would capture the entire market demand and it would earn profit equal

$$\pi_1^* \approx (a - p^*)p^* - (a - p^*)^2. \tag{9}$$

Firm 1 would have no incentives to undercut below p^* if and only if the profit given by (8) is not smaller than the profit given by (9), i.e.,

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[3] This modification is related to the assumption of capacity constraints; compare, e.g., Tirole [1988, pp. 212–213].

$$\frac{a-p^*}{2}p^* - \left(\frac{a-p^*}{2}\right)^2 \geq (a-p^*)p^* - (a-p^*)^2, \tag{10}$$

or

$$(a-p^*)\left(\frac{3}{5}a - p^*\right) \geq 0 \cdot \tag{11}$$

Since $p^* < a$, it follows from (11) that none of the firms will have any incentives to undercut for

$$p^* \leq \frac{3}{5}a \cdot \tag{12}$$

Moreover, in a Nash equilibrium, the profits of firms should not be negative, i.e.,

$$\pi_i^* = \frac{a-p^*}{2}p^* - \left(\frac{a-p^*}{2}\right)^2 \geq 0 \ , \tag{13}$$

or

$$(a-p^*)\left(p^* - \frac{1}{3}a\right) \geq 0 \cdot \tag{14}$$

Hence

$$p^* \geq \frac{1}{3}a \cdot \tag{15}$$

Thus, if p^* is a price at a Nash equilibrium of the above game, then it must belong to the interval $\left[\frac{1}{3}a \ , \ \frac{3}{5}a\right]$. Observe that when both firms set the price equal p^*, none of the firms individually has any incentives to raise its price, because by doing so, according to (7), the firm will earn zero profit. Therefore, any price satisfying (12) and (15) is indeed the Nash equilibrium of the Bertrand game.[4]

However, it should be noticed that the computation of profit according to (7), implicitly assumes that the firm with the lower price will supply the quantity equal to the entire market demand at this price. It would mean that the firm is forced to produce even when its marginal costs are above the price (which happens in any of the above Nash equilibria). Therefore, this assumption is rather unrealistic. The price undercutting firm should be allowed to supply less than the entire market demand, if that is more profitable for that firm.

Thus, the Bertrand model in its standard version is not appropriate for the analysis of oligopolistic price competition in the case of quadratic cost functions. By offering the lowest price in the market, a firm should have an option to sell less than the market demand for its product. This option was

4 Observe that the equilibrium $p^* = \frac{3}{5}a$ Pareto dominates any of the other equilibria.

not important in the case of linear costs, because marginal costs stayed constant no matter the production size. Situation is significantly different when cost functions are quadratic.

The modified model

Let us consider a modified Bertrand duopoly game, in which the firms competing in prices are free to choose their levels of production up to the size of the market demand for their products at the quoted prices. By setting its price at the level of p_i, a profit-maximizing firm i would be interested in producing the output at the level that equalizes the marginal costs ($2q_i$) and the quoted price (p_i), as long as there is enough market demand for its product at the prices set by the firms. Thus firm i will produce $q_i = \frac{p_i}{2}$, as long as the market demand for the product of firm i is not smaller than $\frac{p_i}{2}$.

Formally, we can describe the production decision of firm i as a function of prices as follows. When firm i quotes the lowest price in the market ($p_i < p_j$), it will produce and sell

$$q_i = \min\left\{\frac{p_i}{2}, \mathrm{a} - p_i\right\}.$$

In the case of identical prices set by the suppliers ($p_i = p_j$), the output sold by firm i equals to

$$q_i = \min\left\{\frac{p_i}{2}, \frac{\mathrm{a} - p_i}{2}\right\}.$$

And finally, when firm i quotes the highest price ($p_i > p_j$), the optimal production and sales level is given by[5]

$$\min\left\{\frac{p_i}{2}, \mathrm{a} - \min\left\{\frac{p_j}{2}, \mathrm{a} - p_j\right\} - p_i\right\},$$

as long as the above value is positive. Thus, when $p_i > p_j$, firm i will produce and sell:

$$q_i = \max\left(0,\ \min\left\{\frac{p_i}{2}, \mathrm{a} - \min\left\{\frac{p_i}{2}, \mathrm{a} - p_i\right\} - p_i\right\}\right).$$

Hence the quantity of the product sold by firm i as a function of prices could be summarized as

[5] In this case, we assume the most natural efficient-rationing rule, i.e., the customers with the highest valuation of the product buy first from the cheaper producer. For a discussion of rationing rules see, for example, Tirole [1988, pp. 212–214].

$$q_i(p_i, p_j) = \begin{cases} \min\left\{\frac{p_i}{2}, a - p_i\right\} & \text{for } p_i < p_j \\ \min\left\{\frac{p_i}{2}, \frac{a - p_i}{2}\right\} & \text{for } p_i = p_j \\ \max\left(0, \min\left\{\frac{p_i}{2}, a - \min\left\{\frac{p_i}{2}, a - p_j\right\} - p_i\right\}\right) & \text{for } p_i > p_j \end{cases} \quad (16)$$

Given the above function of firm *i*'s output level, the profit of firm *i* is computed as

$$\pi_i(p_i, p_j) = p_i \; q_i(p_i, p_j) - (q_i(p_i, p_j))^2 . \quad (17)$$

Now, we will focus on the possibility of the existence of a symmetric Nash equilibrium of our modified Bertrand game. In a symmetric equilibrium (if it exists), each firm *i* would be interested in setting the price p_i to maximize

$$\pi_i(p_i, p_i) = p_i \; q_i(p_i, p_i) - (q_i(p_i, p_i))^2 , \quad (18)$$

where $q_i(p_i, p_i)$ is given by (16), i.e., $q_i(p_i, p_i) = \min\left\{\frac{p_i}{2}, \frac{a - p_i}{2}\right\}$. The profit-maximizing price p_i, denoted by p^*, equals

$$p^* = \frac{1}{2}a . \quad (19)$$

At the above price, each firm *i* would produce and sell

$$q_i^* = \frac{1}{4}a \quad (20)$$

earning the profit equal to

$$\pi_i^* = \frac{1}{16}a^2 . \quad (21)$$

Summing up the above considerations, the price $p^* = \frac{1}{2}a$ is the only symmetric strategy profile that maximizes firms profits and clears the market. Thus, it is the only candidate for a symmetric Nash equilibrium of our modified game.

Observe that when both firms charge the same price given by (19) and produce the output given by (20), none of them has any incentives to undercut. The argument is as follows. When firm *j* quotes price $p_j = p^* = \frac{1}{2}a$, it follows from (16) that by charging the price p_i below p_j firm *i* produces and sells the output $q_i = \min\left\{\frac{p_i}{2}, a - p_i\right\}$ which is smaller than $\frac{1}{4}a$. From (18) it follows that the profit of firm i would be lower than $\pi_i^* = \frac{1}{16}a^2$. Thus, none of the firms will be interested in reducing its price below $p^* = \frac{1}{2}a$.

Now, we should consider a possibility for any of the firms to individually increase its price. When firm j quotes price $p_j = p^* = \frac{1}{2}a$, it follows from (16) that by charging the price p_i above p_j firm i produces and sells

$$q_i = max\left(0,\ \min\left\{\frac{p_i}{2}\ , a - \min\left\{\frac{p_j}{2}, a - p_j\right\} - p_i\right\}\right), \tag{22}$$

i.e.,

$$q_i = \begin{cases} \frac{3}{4}a - p_i & for\ \ p_i \le \frac{3}{4}a \\ 0 & for\ \ p_i > \frac{3}{4}a \end{cases} \tag{23}$$

By considering only a small increase of price p_i above $p_j = p^* = \frac{1}{2}a$, we may limit ourselves to the case when $\frac{1}{2}a < p_i \le \frac{3}{4}a$. Then, from (18) it follows that the profit of firm i amounts to

$$\pi_i = p_i\left(\frac{3}{4}a - p_i\right) - \left(\frac{3}{4}a - p_i\right)^2. \tag{24}$$

From (24) we have that the derivative of the above profit function is given as

$$\frac{d\pi_i}{dp_i} = -4p_i + \frac{9}{4}a. \tag{25}$$

Evaluating the derivative (25) at $p_i = \frac{1}{2}a$, we obtain

$$\frac{d\pi_i}{dp_i} = \frac{1}{4}a > 0. \tag{26}$$

Thus, by raising the price above $p^* = \frac{1}{2}a$, firm i can increase its profit above $\pi_i^* = \frac{1}{16}a^2$, i.e. the strategy profile $p_i = p_j = p^* = \frac{1}{2}a$ is not a Nash equilibrium.

Therefore, it follows from the above considerations that our modified Bertrand competition game has no symmetric Nash equilibrium in pure strategies.

Conclusions

In this note, we presented the complication that arises in the simple analysis of the static price competition when the linear cost functions are replaced by the quadratic costs. After adjusting the standard model of Bertrand competition to the new cost environment, we obtain a game with no symmetric equilibrium in pure strategies. Therefore, a solution to the modified static model must involve either asymmetric behavior or randomization. Since we expect identical firms to behave in a similar way, a symmetric equilibrium in mixed strategies should be considered. This conclusion has important consequences for the behavior of firms operating under the quadratic cost functions.

Following the results of our analysis, when the costs of production are described by the quadratic function, we should expect often changes of prices by oligopolistic firms, rather than a stable market situation described by the standard Bertrand model. In order to capture the price fluctuations in the industry characterized by quadratic cost functions mixed strategies must be considered, and the dynamic framework of analysis becomes even more important.

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KONKURENCJA TYPU BERTRANDA PRZY KWADRATOWYCH FUNKCJACH KOSZTÓW

Streszczenie

Głównym celem niniejszej pracy jest zbadanie różnic w rezultatach konkurencji Bertranda, gdy liniowe funkcje kosztów zostaną zastąpione funkcjami kwadratowymi. Udowadniamy, że wprowadzenie kwadratowych funkcji kosztów prowadzi do jakościowej zmiany konkurencji pomiędzy przedsiębiorstwami. W tym przypadku, standardowe założenie, że firma oferująca swoje produkty po niższej cenie jest zainteresowana przejęciem całego rynku, jest nie tylko nierealistyczne (nawet w przypadku braku ograniczeń mocy wytwórczych), ale po prostu nieracjonalne z punktu widzenia maksymalizacji zysku. Zatem wyniki analiz zawartych we wcześniejszej literaturze przedmiotu są mylące. Demonstrujemy, że duopolistyczny model Bertranda powinien zostać zmodyfikowany, aby lepiej uwzględnić zachowanie firm w warunkach kwadratowych funkcji kosztów. W szczególności, uchylamy założenie, że podaż firm jest zdeterminowana przez istniejący popyt. Proponujemy zmodyfikowaną grę duopolu typu Bertranda, w której firmy konkurujące cenowo mogą swobodnie wybierać poziom produkcji nieprzekraczający wielkości popytu rynkowego na ich produkty przy danych cenach. Przy zmodyfikowanych założeniach, udowadniamy, że gra opisująca statyczną konkurencję cenową identycznych duopolistów nie posiada symetrycznej równowagi w strategiach czystych. Wyniki naszej analizy prowadzą do wniosku, że przy kwadratowych funkcjach kosztów powinniśmy spodziewać się raczej fluktuacji cen na rynkach oligopolistycznych niż sytuacji stabilnej równowagi opisanej w standardowych modelach konkurencji cenowej.

Słowa kluczowe: konkurencja typu Bertranda, kwadratowe funkcje kosztów, statyczna konkurencja cenowa

Kody klasyfikacji JEL: L13, L41, O31