Załącznik 1. Pełne wyprowadzenie modelu

Oznaczenia zmiennych w załączniku są zgodne z oznaczeniami w publikacji. Bardziej szczegółowe informacje dotyczące stosowanego aparatu matematycznego można znaleźć np. w [de la Fuente, 2000: 566–580, 622–625][1], [Kłopotowski, Kołatkowski, 2007: 116–189][2] i [Dorosiewicz, 2003: 9–40][3].

Założenia modelu

Funkcja użyteczności pojedynczego gospodarstwa domowego:

$$u_i(c) = \int_0^{\infty} \left(\frac{(c_i(t))^{1-\theta}}{1-\theta} \right) e^{-\rho t} dt \qquad \text{(Z1)}$$

Zagregowana funkcja produkcji w gospodarce:

$$Y(t) = Y_1(t) + Y_2(t) = A(t)(x(t)K(t))^{\alpha}(H_1(t))^{1-\alpha} + A(t)((1-x(t))K(t))^{\alpha}(H_2(t))^{1-\alpha} \qquad \text{(Z2)}$$

Tempo wzrostu w gospodarce zgodne ze zmodyfikowanym modelem Nelsona-Phelpsa:

$$\frac{\dot{A}(t)}{A(t)} = g(H(t)) + \Phi(H(t)) \left(\frac{T_m(t) - A_i(t)}{A_i(t)} \right) \qquad \text{(Z3)}$$

Tempo wzrostu w gospodarce kraju najbardziej rozwiniętego. Jednocześnie zakłada się, że $g(H(t)) \leq g$:

$$\frac{\dot{T}_m(t)}{T_m(t)} = g \qquad \text{(Z4)}$$

Zagregowany kapitał i kapitał ludzki w gospodarce wynosi:

$$K(t) = \int_0^1 k_i(t)di = \int_0^{\vartheta} k_1(t)di + \int_0^{1-\vartheta} k_2(t)di = K_1(t) + K_2(t) \qquad \text{(Z5)}$$

$$H(t) = \int_0^1 h_i(t)di = \int_0^{\vartheta} h_1(t)di + \int_0^{1-\vartheta} h_2(t)di = H_1(t) + H_2(t)$$

Czynniki produkcji wynagradzane są swoją krańcową produktywnością, zatem:

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1 Można tam znaleźć w szczególności informacje o metodzie hamiltonianów i jej zastosowaniu w rozwiązywaniu modelu Ramseya-Cassa-Koopmansa.

2 Elementy wyznaczania ekstremów funkcji, liczenia pochodnych i całek.

3 Rozwiązywanie równań różniczkowych.

$$R_k(t) = \frac{\alpha}{K(t)} Y(t) \tag{Z6}$$

$$R(t,\ H_1(t)) = \frac{(1-\alpha)}{H_1(t)} Y_1$$

$$R(t,\ H_2(t)) = \frac{(1-\alpha)}{H_2(t)} Y_2$$

Z maksymalizacji zysków przez firmy:

$$\frac{dY(t)}{dx(t)} = 0 \Rightarrow \alpha A(t)(K(t))^{\alpha}((x(t))^{\alpha-1}\left(H_1(t))^{(1-\alpha)} - (1-x(t))^{\alpha-1} H_2(t)^{(1-\alpha)}\right) = 0 \tag{Z7}$$

$$\underset{A(t),\ K(t),\alpha\neq 0}{\Rightarrow} (x(t))^{\alpha-1}(H_1(t))^{(1-\alpha)} = (1-x(t))^{\alpha-1}(H_2(t))^{(1-\alpha)}$$

$$\underset{0<x<1;\alpha\langle 1;H_1(t),H_2(t)\rangle 0}{\Rightarrow} (\frac{(1-x(t))}{x(t)})^{(1-\alpha)} = \left(\frac{H_2(t)}{H_1(t)}\right)^{(1-\alpha)} \Rightarrow \frac{(1-x(t))}{x(t)} = \left(\frac{H_2(t)}{H_1(t)}\right)$$

$$\Rightarrow x(t) = \left(\frac{H_1(t)}{H_2(t)}\right) - x(t) * \left(\frac{H_1(t)}{H_2(t)}\right) \Rightarrow x(t) = \frac{\left(\frac{H_1(t)}{H_2(t)}\right)}{\left(1+\left(\frac{H_1(t)}{H_2(t)}\right)\right)}$$

$$\Rightarrow x(t) = \frac{H_1(t)}{H_2(t) \ + H_1(t)}$$

Wobec tego wynagrodzenie kapitału ludzkiego wynosi:

$$R(H_1,\ t) = (1-\alpha)A(t)K(t)^{\alpha} H_1(t)^{-\alpha} * \left(\frac{H_1(t)}{H_2(t)+H_1(t)}\right)^{\alpha} = \tag{Z8}$$

$$= (1-\alpha)A(t)K(t)^{\alpha} * \left(\frac{1}{H_2(t)+H_1(t)}\right)^{\alpha} =$$

$$= (1-\alpha)A(t)K(t)^{\alpha} H_2(t)^{-\alpha} * \left(\frac{H_2(t)}{H_2(t)+H_1(t)}\right)^{\alpha} = R(H_2,t) = R_H(t) =$$

$$= (1-\alpha)A(t)K(t)^{\alpha} * \frac{H(t)^{1-\alpha}}{H(t)} = \frac{(1-\alpha)}{H(t)} Y$$

Zauważmy też, że ogólnie mamy:

$$H_1(t)^{1-\alpha} * x(t)^{\alpha} + H_2(t)^{1-\alpha} * (1 - x(t))^{\alpha} = \quad \text{(Z9)}$$

$$= H_1(t)^{1-\alpha} * \left(\frac{H_1(t)}{H_2(t) + H_1(t)}\right)^{\alpha} + H_2(t)^{1-\alpha} * \left(\frac{H_2(t)}{H_2(t) + H_1(t)}\right)^{\alpha} = \frac{H_1(t) + H_2(t)}{(H_2(t) + H_1(t))^{\alpha}} = H(t)^{1-\alpha}$$

Ograniczenie budżetowe pojedynczego gospodarstwa domowego:

$$(1 + \tau_C) * c_i(t) + \varepsilon_i(t) + \iota_i(t) + \tau(t) = (1 - \tau_D) * (R_K(t) * k_i(t) + R_H(t) * h_i(t)) \quad \text{(Z10)}$$

Gdzie $\iota_i(t)$ wydatki inwestycyjne gospodarstwa domowego na kapitał rzeczowy. Zmiana zasobu kapitału dla pojedynczego gospodarstwa domowego odbywa się zgodnie ze wzorem:

$$\dot{k}(t) = -\sigma k_i(t) + \iota_i(t) \quad \text{(Z11)}$$

Zatem:

$$\dot{k}_i(t) = (1 - \tau_D) * (R_K(t) * k_i(t) + R_H(t) * h_i(t)) - (1 + \tau_C) c_i(t) - \varepsilon_i(t) - \sigma k_i(t) - \tau(t) \quad \text{(Z12)}$$

Zmiana zasobu kapitału ludzkiego dla pojedynczego gospodarstwa domowego odbywa się zgodnie ze wzorem:

$$\dot{h}_i(t) = -\sigma_H h_i(t) + \left(\varepsilon(t)^{\gamma} * (b + \tau(t) + tr(t))^{1-\gamma}\right)^{\varphi} \quad \text{(Z13)}$$

Tempo wzrostu podatku ryczałtowego jest ograniczone w t = 0:

$$\frac{\dot{\tau}(t)}{\tau(t)} \leq g(H(0)) \quad \text{(Z14)}$$

Ograniczenie budżetowe dla transferów na edukację finansowanych podatkami nieneutralnymi względem cen:

$$\int_0^1 tr \; di = \tau_D \int_0^1 (R_K k_i) + (R_H(H_i) h_i) di + \tau_C \int_0^1 c_i di \quad \text{(Z15)}$$

Jednocześnie celem państwa jest maksymalizacja użyteczności bogatszej części społeczeństwa:

$$\max_{\tau_C, \tau_D} U(\tau_C, \tau_D) = \int_0^{\vartheta} u_i(\tau_C, \tau_D) di \quad \text{(Z16)}$$

Rozwiązanie problemu konsumenta

Dla każdego gospodarstwa domowego można zapisać hamiltonian:

$$\mathcal{H}_t^{c,i} = \frac{(c_i(t))^{1-\theta}}{1-\theta} + \lambda_1(t) * \tag{Z17}$$

$$*((R_k(t) * k_i(t) + R_H(t) * h_i(t)) * (1-\tau_D) - (1+\tau_c) * c_i(t) - \varepsilon_i(t) - \sigma_K * k_i(t) - \tau(t)) +$$

$$+\lambda_2(t) * (-\sigma_H * h_i(t) + \left(\varepsilon_i(t)^{\gamma} * (b + \tau(t) + tr(t))^{1-\gamma}\right)^{\varphi}$$

Otrzymujemy warunki maksymalizacji:

$$\begin{cases} \lambda_1(t) = \dfrac{(c_i(t))^{-\theta}}{(1+\tau_c)} & (1') \\ \lambda_1(t) = \lambda_2(t) * \varphi * \gamma * (\varepsilon_i(t))^{(\varphi*\gamma-1)} * (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} & (2') \\ -\lambda_1(t)(R_k(t) * (1-\tau_D) - \sigma_K) = \dot{\lambda}_1(t) - \rho * \lambda_1(t) & (3') \\ -(\lambda_1(t) R_H(t) * (1-\tau_D) - \lambda_2(t) * \sigma_H) = \dot{\lambda}_2(t) - \rho * \lambda_2(t) & (4') \\ \text{warunki transwersalnosci} & (5') \end{cases} \tag{Z18}$$

Z (1') i (3') można uzyskać równanie Eulera dla konsumpcji. Z (3'):

$$-\lambda_1(t)(R_k(t) * (1-\tau_D) - \sigma_K) = \dot{\lambda}_1(t) - \rho * \lambda_1(t) \Rightarrow -(R_k(t) * (1-\tau_D) - \sigma_K) = \tag{Z19}$$

$$= \frac{\dot{\lambda}_1(t)}{\lambda_1(t)} - \rho \Rightarrow \frac{\dot{\lambda}_1(t)}{\lambda_1(t)} = \sigma_K + \rho - R_k(t) * (1-\tau_D)$$

Z (1'):

$$\lambda_1(t) = \frac{(c_i(t))^{-\theta}}{(1+\tau_c)} \Rightarrow \dot{\lambda}_1(t) = -\theta * \frac{(c_i(t))^{-\theta-1}}{(1+\tau_c)} * \dot{c}(t) \Rightarrow \frac{\dot{\lambda}_1(t)}{\lambda_1(t)} = \tag{Z20}$$

$$= -\theta * \frac{\dot{c}_i(t)}{c_i(t)} \overset{\text{podstawiajac (24)}}{\Rightarrow} -\theta * \frac{\dot{c}_i(t)}{c_i(t)} = \sigma_K + \rho - R_k(t) \Rightarrow$$

$$\frac{\dot{c}_i(t)}{c_i(t)} = \frac{-\sigma_K - \rho + R_k(t) * (1-\tau_D)}{\theta}$$

Zatem:

$$c_i(t) = c(0) * e^{\frac{(-\sigma_K - \rho)t + \int_0^t (R_k(T)*(1-\tau_D))dT}{\theta}} \tag{Z21}$$

Analogicznie z (2'), (3'), (4') można wyznaczyć $\varepsilon_i(t)$. Z (2'):

$$\lambda_1(t) = \lambda_2(t) * \varphi * \gamma * (\varepsilon_i(t))^{(\varphi*\gamma-1)} * (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} \quad \Rightarrow \tag{Z22}$$

$$\lambda_2(t) = \lambda_1(t) * \frac{1}{\gamma * \varphi} * (\varepsilon_i(t))^{1-\varphi*\gamma} * (b + tr(t) + \tau(t))^{(\gamma-1)*\varphi} \Rightarrow$$

$$\dot{\lambda}_2(t) = \dot{\lambda}_1(t) * \frac{1}{\gamma * \varphi} * (\varepsilon_i(t))^{1-\varphi*\gamma} * (b + tr(t) + \tau(t))^{(\gamma-1)*\varphi} + \lambda_1(t) * \frac{1 - \gamma * \varphi}{\gamma * \varphi} *$$

$$* (\varepsilon_i(t))^{-\varphi*\gamma} * \dot{\varepsilon}_i(t) * (b + tr(t) + \tau(t))^{(\gamma-1)*\varphi} + \frac{\gamma - 1}{\gamma} * \lambda_1(t) * (\varepsilon_i(t))^{1-\varphi*\gamma} \; *$$

$$* (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi-1} (t\dot{r}(t) + \dot{\tau}(t)) \Rightarrow$$

$$\frac{\dot{\lambda}_2(t)}{\lambda_2(t)} = \frac{\dot{\lambda}_1(t)}{\lambda_1(t)} + (1 - \varphi * \gamma) * \frac{\dot{\varepsilon}_i(t)}{\varepsilon_i(t)} + \varphi(\gamma - 1) * \frac{(t\dot{r}(t) + \dot{\tau}(t))}{b + tr(t) + \tau(t)}$$

Z (4'):

$$-(\lambda_1(t) R_H(t) * (1 - \tau_D) - \lambda_2(t) * \sigma_H) = \dot{\lambda}_2(t) - \rho * \lambda_2(t) \overset{(2')}{\Rightarrow} \tag{Z23}$$

$$-(\lambda_2(t) * \varphi * \gamma * (\varepsilon_i(t))^{(\varphi*\gamma-1)} * (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} * R_H(t) * (1 - \tau_D) - \lambda_2(t) * \sigma_H)$$

$$= \dot{\lambda}_2(t) - \rho * \lambda_2(t) \Rightarrow \frac{\dot{\lambda}_2(t)}{\lambda_2(t)} = \rho + \sigma_H - \varphi * \gamma * (\varepsilon_i(t))^{(\varphi*\gamma-1)} *$$

$$* (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} * R_H(t) * (1 - \tau_D)$$

Przyrównując (Z22) i (Z23) przy jednoczesnym podstawieniu (Z19)[4]:

$$\sigma_K + \rho - R_k(t) * (1 - \tau_D) + (1 - \varphi * \gamma) * \frac{\dot{\varepsilon}_i(t)}{\varepsilon_i(t)} - \varphi(\gamma - 1) * \frac{(t\dot{r}(t) + \dot{\tau}(t))}{b + tr(t) + \tau(t)} = \tag{Z24}$$

$$= \rho + \sigma_H - \varphi * \gamma * (\varepsilon_i(t))^{(\varphi*\gamma-1)} * (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} * R_H(t) * (1 - \tau_D) \Rightarrow$$

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4 Z założenia $1 - \gamma > 0$

$$\frac{\dot{\varepsilon}_i(t)}{\varepsilon_i(t)} = \frac{1}{(1-\varphi*\gamma)} * \left(R_k(t)*(1-\tau_D) + \varphi(\gamma-1)*\frac{(t\dot{r}(t)+\dot{\tau}(t))}{b+tr(t)+\tau(t)} \right) -$$

$$-\frac{1}{(1-\varphi*\gamma)} * \varphi*\gamma*(b+tr(t)+\tau(t))^{(1-\gamma)*\varphi} * R_H(t)*(1-\tau_D)*(\varepsilon_i(t))^{(\varphi*\gamma-1)}$$

Zatem:

$$\dot{\varepsilon}_i(t) = \frac{\varepsilon_i(t)}{(1-\varphi*\gamma)} * \left(R_k(t)*(1-\tau_D) + \varphi(\gamma-1)*\frac{(t\dot{r}(t)+\dot{\tau}(t))}{b+tr(t)+\tau(t)} \right) - \tag{Z25}$$

$$-\frac{1}{(1-\varphi*\gamma)} * \varphi*\gamma*(b+tr(t)+\tau(t))^{(1-\gamma)*\varphi} * R_H(t)*(1-\tau_D)*(\varepsilon_i(t))^{\varphi*\gamma}$$

W celu rozwiązania tego równania różniczkowego podstawiamy:

$$\mu(t) = (\varepsilon_i(t))^{1-\varphi*\gamma} \tag{Z26}$$

Otrzymujemy:

$$\dot{\mu}(t) = \left(R_k(t)*(1-\tau_D) + \varphi(\gamma-1)*\frac{(t\dot{r}(t)+\dot{\tau}(t))}{b+tr(t)+\tau(t)} \right) * \mu(t) + \tag{Z27}$$

$$-\varphi*\gamma*(b+tr(t)+\tau(t))^{(1-\gamma)*\varphi} * R_H(t)*(1-\tau_D)$$

Jest to liniowe równanie różniczkowe pierwszego rzędu. Jego rozwiązanie to (dla części jednorodnej):

$$\dot{\mu}(t) = \left(R_k(t)*(1-\tau_D) + \varphi(\gamma-1)*\frac{(t\dot{r}(t)+\dot{\tau}(t))}{b+tr(t)+\tau(t)} \right) * \mu(t) \Rightarrow$$

$$\frac{\dot{\mu}(t)}{\mu(t)} = \left(R_k(t)*(1-\tau_D) + \varphi(\gamma-1)*\frac{(t\dot{r}(t)+\dot{\tau}(t))}{b+tr(t)+\tau(t)} \right) \Rightarrow$$

$$\mu(t) = s * e^{\int_0^t \left(R_k(\xi)*(1-\tau_D) + \varphi(\gamma-1)*\frac{(t\dot{r}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi}$$

Uzmienniając stałą:

$$\mu(t) = s(t) * e^{\int_0^t \left(R_k(\xi)*(1-\tau_D) + \varphi(\gamma-1)*\frac{(t\dot{r}(t)+\dot{\tau}(t))}{b+tr(t)+\tau(t)} \right) d\xi} \Rightarrow$$

$$\dot{s}(t) * e^{\int_0^t \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(t)+\dot{\tau}(t))}{b+tr(t)+\tau(t)} \right) d\xi} =$$

$$= -\varphi * \gamma * (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} * R_H(t) * (1-\tau_D) \Rightarrow$$

$$s(t) = \left(\int_0^t [(-\varphi * \gamma * (b + tr(\zeta) + \tau(\zeta))^{(1-\gamma)*\varphi} * R_H(\zeta) * (1-\tau_D)) * \right.$$

$$\left. * e^{-\int_0^\zeta \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi}] d\zeta \right) + \text{constans}$$

Zatem:

$$\mu(t) = \left(\mu(0) - \left(\int_0^t [(\varphi * \gamma * (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} * R_H(t) * (1-\tau_D)) * \right. \right. \quad \text{(Z28)}$$

$$\left. \left. * e^{-\int_0^\zeta \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi}] d\zeta \right) \right) *$$

$$* e^{\int_0^t \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi}$$

Cofając podstawienie, otrzymuje się:

$$\varepsilon_i(t) = e^{\left(\int_0^t \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi \right)/(1-\varphi*\gamma)} * \quad \text{(Z29)}$$

$$\left(\varepsilon(0)^{1/(1-\varphi*\gamma)} - \left(\int_0^t (\varphi * \gamma * (b + tr(\zeta) + \tau(\zeta))^{(1-\gamma)*\varphi} * R_H(\zeta) * (1-\tau_D)) * \right. \right.$$

$$\left. \left. * e^{-\int_0^\zeta \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi} d\zeta \right) \right)^{1/(1-\varphi*\gamma)}$$

Z równania (Z9):

$$\dot{k}_i(t) = (1-\tau_D) * (R_K(t) * k_i(t) + R_H(t) * h_i(t)) - (1+\tau_C) * c_i(t) - \varepsilon_i(t) - \quad \text{(Z30)}$$

$$-\sigma_K k(t) - \tau(t) \Rightarrow k_i(t) = e^{-\sigma_K * t + (1-\tau_D) \int_0^t R_K(q) dq} * (k_i(0) +$$

$$+\int_0^t [e^{\sigma_K*\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)R_H(\kappa)h_i(\kappa)-(1+\tau_C)c_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa)]d\kappa)$$

Analogicznie z (Z10):

$$\dot{h}_i(t)=-\sigma_H h(t)+\left(\varepsilon_i(t)^\gamma*(b+\tau+tr(t))^{1-\gamma}\right)^\varphi \Rightarrow \tag{Z31}$$

$$h_i(t)=e^{-\sigma_H t}*(h(0)+\int_0^t e^{\sigma_H\pi}*\left(\varepsilon_i(\pi)^\gamma*(b+\tau+tr(\pi))^{1-\gamma}\right)^\varphi d\pi)$$

Następnie korzysta się z warunków transwersalności:

$$\lim_{t\to\infty} k_i(t)*\lambda_1*e^{-\rho t}=0 \tag{Z32}$$

$$\lim_{t\to\infty} h_i(t)*\lambda_2*e^{-\rho t}=0 \tag{Z33}$$

Z (Z33) podstawiając (Z30) i (2'):

$$\lim_{t\to\infty} h_i(t)*\lambda_2*e^{-\rho t}=\lim_{t\to\infty} e^{-\rho t}*e^{-\sigma_H t}* \tag{Z34}$$

$$*(h_i(0)+\int_0^t e^{\sigma_H\pi}*\left(\varepsilon_i(\pi)^\gamma*(b+\tau+tr(\pi))^{1-\gamma}\right)^\varphi d\pi)*\lambda_1(t)*\frac{1}{\gamma*\varphi}*\varepsilon_i(t)^{1-\varphi*\gamma}*$$

$$*(b+tr(t)+\tau(t))^{(\gamma-1)*\varphi}=0$$

Dzielimy obie strony przez $\frac{1}{\gamma*\varphi}$ i podstawiamy (1'), (Z21) oraz (Z28)[5]:

$$\lim_{t\to\infty} e^{-\rho t}*e^{-\sigma_H t}*(h_i(0)+\int_0^t e^{\sigma_H\pi}*\left((\varepsilon_i(\pi))^\gamma*(b+\tau+tr(\pi))^{1-\gamma}\right)^\varphi d\pi)* \tag{Z35}$$

$$*(b+tr(t)+\tau(t))^{(\gamma-1)*\varphi}*\frac{(c_i(t))^{-\theta}}{(1+\tau_c)}*$$

$$*\left(\mu(0)-\left(\int_0^t(\varphi*\gamma*(b+tr(\zeta)+\tau(\zeta))^{(1-\gamma)*\varphi}*R_H(\zeta)*(1-\tau_D))*\right.\right.$$

$$\left.\left.*e^{-\int_0^\zeta\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi}d\zeta\right)\right)*$$

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5 Zauważmy, że $\varepsilon_i(t)^{1-\varphi*\gamma}=\mu(t)$.

$$*e^{\int_0^t\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(t)+\tau(t)}\right)d\xi} = 0 \Rightarrow$$

$$\lim_{t\to\infty} e^{-(\sigma_K+\rho)*t+\int_0^t\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi} * (b+tr(t)+\tau(t))^{(\gamma-1)*\varphi} *$$

$$*\left(\mu(0)-\left(\int_0^t(\varphi*\gamma*(b+tr(\zeta)+\tau(\zeta))^{(1-\gamma)*\varphi}*R_H(\zeta)*(1-\tau_D))*\ d\zeta\right)\right)*$$

$$*e^{-\int_0^\zeta\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi}$$

$$*(h_i(0)+\int_0^t e^{\sigma_H\pi}*\left(\varepsilon(\pi)^\gamma*(b+\tau+tr(\pi))^{1-\gamma}\right)^\varphi d\pi)*(c_i(0))^{-\theta}*e^{(\sigma_K+\rho)t-\int_0^t(R_k(T)*(1-\tau_D))dT} = 0$$

więc[6]:

$$c_i(0)^{-\theta}*\lim_{t\to\infty}(b+tr(t)+\tau(t))^{(\gamma-1)*\varphi}*e^{\int_0^t\left(\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi} * \tag{Z36}$$

$$*\left(\mu(0)-\left(\int_0^t(\varphi*\gamma*(b+tr(\zeta)+\tau(\zeta))^{(1-\gamma)*\varphi}*R_H(\zeta)*(1-\tau_D))*\right.\right.$$

$$\left.\left.*e^{-\int_0^\zeta\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi} d\zeta\right)\right) = 0$$

Zakładając, że $w(t) = tr(t) + \tau(t)$ jest funkcją niemalejącą[7], to (Z36) jest równoznaczne z[8]:

$$\mu(0)-\left(\int_0^\infty(\varphi*\gamma*(b+tr(t)+\tau(t))^{(1-\gamma)*\varphi}*R_H(t)*(1-\tau_D))*\right. \tag{Z37}$$

$$\left.*e^{-\int_0^t\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi} dt\right) = 0 \Rightarrow$$

6 $c_i(0)$ nie zmienia się z czasem, więc możemy wyciągnąć przed granicę.

7 Zauważmy, że $\tau(t)$ rośnie z wcześniej przyjętej definicji, a $tr(t)$ maleje wtedy i tylko wtedy, gdy spada dochód lub spada konsumpcja w gospodarce *ceteris paribus*, co w tej gospodarce nie występuje.

8 c(0) > 0.

$$\int_0^\infty (\varphi * \gamma * (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} * R_H(t) * (1-\tau_D)) * e^{-\int_0^t \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi} dt = \mu(0)$$

więc:

$$\mu(t) = (\int_0^\infty (\varphi * \gamma * (b + tr(t) + \tau(t))^{(1-\gamma)*\varphi} * R_H(t) * (1-\tau_D)) * \tag{Z38}$$

$$* e^{-\int_0^t \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi} dt -$$

$$-\left(\int_0^t (\varphi * \gamma * (b + tr(\zeta) + \tau(\zeta))^{(1-\gamma)*\varphi} * R_H(\zeta) * (1-\tau_D)) * e^{-\int_0^\zeta \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi} d\zeta \right))$$

$$* e^{\int_0^t \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi} =$$

$$= e^{\int_0^t \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi} * \int_t^\infty (\varphi * \gamma * (b + tr(T) + \tau(T))^{(1-\gamma)*\varphi} * R_H(T) * (1-\tau_D)) *$$

$$* e^{-\int_0^T \left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)} \right) d\xi} dT$$

Z (Z32) podstawiając (1’), (Z30):

$$\lim_{t\to\infty} k(t) * \lambda_1 * e^{-\rho t} = \lim_{t\to\infty} e^{-\rho t} * \frac{(c_i(t))^{-\theta}}{(1+\tau_c)} * e^{-\sigma_K * t + (1-\tau_D)\int_0^t R_K(q)dq} * (k(0) + \tag{Z39}$$

$$+ \int_0^t (e^{\sigma_K * \kappa - (1-\tau_D)\int_0^\kappa R_K(q)dq} * ((1-\tau_D) * R_H(\kappa) * h_i(\kappa) - (1+\tau_C) * c_i(\kappa) - \varepsilon_i(\kappa) - \tau(\kappa)))d\kappa)$$

Podstawiając (Z21):

$$(c_i(t))^{-\theta} = \left(c_i(0) * e^{\frac{(-\sigma_K - \rho)t + \int_0^t (R_k(T)*(1-\tau_D))dT}{\theta}} \right)^{-\theta} = c_i(0) * e^{(\sigma_K + \rho)t - \int_0^t (R_k(T)*(1-\tau_D))dT} \tag{Z40}$$

Łącząc (Z39) i (Z40), otrzymuje się:

$$\lim_{t\to\infty} k_i(t) * \lambda_1 * e^{-\rho t} = \lim_{t\to\infty} \frac{c_i(0)}{(1+\tau_c)} * (k_i(0) + \tag{Z41}$$

$$+\int_0^t e^{\sigma_K*\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-(1+\tau_C)*c_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))d\kappa)=0$$

$$k_i(0)+\int_0^\infty (e^{\sigma_K*t-(1-\tau_D)\int_0^t R_K(q)dq}* \tag{Z42}$$

$$*((1-\tau_D)*R_H(t)*h_i(t)-(1+\tau_C)*c_i(t)-\varepsilon_i(t)-\tau(t)))dt=0$$

wyciągając $\int_0^\infty e^{\sigma_K*t-(1-\tau_D)\int_0^t R_K(q)dq}*(-(1+\tau_C)*c_i(t))dt$ i podstawiając (Z21):

$$\int_0^\infty e^{\sigma_K*t-(1-\tau_D)\int_0^t R_K(q)dq}*c_i(t)\ dt=\frac{1}{(1+\tau_C)}*(k_i(0)+ \tag{Z43}$$

$$+\int_0^\infty e^{\sigma_K*t-(1-\tau_D)\int_0^t R_K(q)dq}*((1-\tau_D)*R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t))dt)\Rightarrow$$

$$\int_0^\infty e^{\sigma_K*t-(1-\tau_D)\int_0^t R_K(q)dq}*c_i(0)*e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}\quad dt=$$

$$=\frac{1}{(1+\tau_C)}*(k_i(0)+$$

$$+\int_0^\infty\left(e^{\sigma_K*t-(1-\tau_D)\int_0^t R_K(q)dq}*((1-\tau_D)*R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t))\right)dt)$$

Zatem c(0) wynosi:

$$c_i(0)=\frac{k_i(0)+\int_0^\infty\left(e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq}*((1-\tau_D)*R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t))\right)dt}{(1+\tau_C)*\int_0^\infty e^{\sigma t-(1-\tau_D)\int_0^t R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}\quad dt} \tag{Z44}$$

Podstawiając (Z21) i (Z44) do funkcji użyteczności konsumenta

$$U_i=\frac{1}{1-\theta}* \tag{Z45}$$

$$*\int_0^\infty \left(\frac{k_i(0)+\int_0^t e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))d\kappa}{(1+\tau_C)*\int_0^\infty e^{\sigma_K*\varsigma-(1-\tau_D)\int_0^\varsigma R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)\varsigma+\int_0^\varsigma(R_k(T)*(1-\tau_D))dT}{\theta}}d\varsigma}\right)^{1-\theta}*$$

$$*e^{-\rho t+\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}}dt)$$

Korzyści z wprowadzenia podatku dochodowego

Przepisujemy problem państwa jako:

$$\max_{\tau_D,\tau_C} U(\tau_D,\tau_C) \tag{Z46}$$

Z równania (Z45) mamy

$$\frac{dU(\tau_D,\tau_C)}{d\ \tau_D}=\frac{1}{1-\theta}* \tag{Z47}$$

$$*\int_0^\infty[\frac{d}{d\ \tau_D}((\frac{k_i(0)+\int_0^t e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))d\kappa}{(1+\tau_C)*\int_0^\infty e^{\sigma_K*\varsigma-(1-\tau_D)\int_0^\varsigma R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)\varsigma+\int_0^\varsigma(R_k(T)*(1-\tau_D))dT}{\theta}}d\varsigma})^{1-\theta}*$$

$$*e^{-\rho t+\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}}))]dt$$

Ze wzoru na pochodną iloczynu otrzymujemy:

$$\frac{dU(\tau_D,\tau_C)}{d\ \tau_D}=\frac{1}{1-\theta}* \tag{Z48}$$

$$*\int_0^\infty[(\frac{d}{d\ \tau_D}\left(\left(\frac{k_i(0)+\int_0^t e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))d\kappa}{(1+\tau_C)*\int_0^\infty e^{\sigma_K*\varsigma-(1-\tau_D)\int_0^\varsigma R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)\varsigma+\int_0^\varsigma(R_k(T)*(1-\tau_D))dT}{\theta}}d\varsigma}\right)^{1-\theta}\right)*$$

$$*e^{\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}}+\frac{d\left(e^{\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}}\right)}{d\ \tau_D}*$$

$$*\left(\frac{k_i(0)+\int_0^t e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))d\kappa}{(1+\tau_C)*\int_0^\infty e^{\sigma_K*\varsigma-(1-\tau_D)\int_0^\varsigma R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^\varsigma(R_k(T)*(1-\tau_D))dT}{\theta}}d\varsigma}\right)^{1-\theta})*e^{-\rho t}]dt$$

Podstawiając

$$e^{\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}} = \Delta(t) \qquad (Z49)$$

otrzymuje się:

$$\frac{dU(\tau_D,\tau_C)}{d\ \tau_D} = \frac{1}{1-\theta} * \int_0^\infty e^{-\rho t} * (\frac{d(\Delta(t))}{d\ \tau_D} * (c(0))^{1-\theta} + \Delta(t) * \frac{d\left((c(0))^{1-\theta}\right)}{d\ \tau_D})\ dt \qquad (Z50)$$

$$\frac{d(\Delta(t))}{d\ \tau_D} = \frac{d\left(e^{\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}}\right)}{d\ \tau_D} = \qquad (Z51)$$

$$= \Delta(t) * \frac{d\left(\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}\right)}{d\ \tau_D} =$$

$$= \frac{\Delta(t)*(1-\theta)}{\theta} * \frac{d(\int_0^t(R_k(M)*(1-\tau_D))dM)}{d\ \tau_D} =$$

$$= \frac{\Delta(t)*(1-\theta)}{\theta} * \int_0^t \frac{d(R_k(M)*(1-\tau_D))}{d\ \tau_D} dM$$

Wykorzystując założenie o $\tau_D = 0$, otrzymuje się:

$$\frac{d(\Delta(t))}{d\ \tau_D} = \frac{\Delta(t)*(1-\theta)}{\theta} * \int_0^t \frac{d(R_k(M))}{d\ \tau_D} - R_k(M)dM \qquad (Z52)$$

$$\frac{d\ ((c(0))^{1-\theta})}{d\ \tau_D} = (1-\theta)*(c(0))^{-\theta} * \frac{d\ (c(0))}{d\ \tau_D} = (1-\theta)*(c(0))^{-\theta} * \qquad (Z53)$$

$$* \frac{d\left(\frac{k_i(0)+\int_0^\infty e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq} * ((1-\tau_D)*R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t))dt}{(1+\tau_C)*\int_0^\infty e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq} * e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}\ dt}\right)}{d\ \tau_D}$$

więc:

$$\frac{d\left(\left(c(0)\right)^{1-\theta}\right)}{d\ \tau_D}=\frac{(1-\theta)*c(0)^{-\theta}}{(1+\tau_C)\left(\int_0^{\infty}e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}dt\right)^2}* \quad \text{(Z54)}$$

$$*(\frac{d\ (k_i(0)+\int_0^{\infty}e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq}*((1-\tau_D)*R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t))dt}{d\ \tau_D}*$$

$$*(1+\tau_C)(\int_0^{\infty}e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}dt)-$$

$$-\frac{d(\int_0^{\infty}e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}dt}{d\ \tau_D}*$$

$$*\left(k_i(0)+\int_0^{\infty}e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq}*((1-\tau_D)*R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t))dt\right))$$

Podstawiając:

$$\frac{\theta}{c(0)*(1+\tau_C)\left(\int_0^{\infty}e^{\sigma t-(1-\tau_D)\int_0^t R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}dt\right)^2}=\nu_1 \quad \text{(Z55)}$$

$$\int_0^{\infty}e^{\sigma t-(1-\tau_D)\int_0^t R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}dt)=\nu_2$$

$$\left(k_i(0)+\int_0^{\infty}e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq}*((1-\tau_D)*R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t))dt\right)=\nu_3$$

$$\frac{(1-\theta)*(c(0))^{-\theta}}{(1+\tau_C)\left(\int_0^{\infty}e^{\sigma t-(1-\tau_D)\int_0^t R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^t(R_k(T)*(1-\tau_D))dT}{\theta}}dt\right)^2}=\nu_4$$

otrzymuje się:

$$\frac{d\left((c(0))^{1-\theta}\right)}{d\ \tau_D} = \nu_4 * \tag{Z56}$$

$$*\left(\frac{d\ (k_i(0) + \int_0^{\infty} e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq} *((1-\tau_D) * R_H(t) * h_i(t) - \varepsilon_i(t) - \tau(t))dt}{d\ \tau_D} *\right.$$

$$*\nu_2 - \frac{d(\int_0^{\infty} e^{\sigma t-(1-\tau_D)\int_0^t R_K(q)dq} * e^{\frac{(-\sigma_K-\rho)t+\int_0^t (R_k(T)*(1-\tau_D))dT}{\theta}} dt)}{d\ \tau_D} * \nu_3) = \nu_4 * (\nu_2 *$$

$$*\int_0^{\infty} \frac{d(e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq} *((1-\tau_D) * R_H(t) * h_i(t) - \varepsilon_i(t) - \tau(t)))}{d\ \tau_D} dt -$$

$$-\nu_3 * \int_0^{\infty} \frac{d(e^{\frac{(-\rho)t-(1-\theta)*(\sigma_K t - \int_0^t (R_k(T)*(1-\tau_D))dT}{\theta}})}{d\ \tau_D} dt$$

$$\frac{d(e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq} *((1-\tau_D) * R_H(t) * h_i(t) - \varepsilon_i(t) - \tau(t)))}{d\ \tau_D} = \tag{Z57}$$

$$= e^{\sigma_K t - \int_0^t R_K(q)dq} *((R_H(t) * h_i(t) - \varepsilon_i(t) - \tau(t)) * \left(\int_0^t R_K(q) - \frac{d(R_K(q))}{d\ \tau_D} dq\right) -$$

$$-\frac{d(\varepsilon_i(t))}{d\ \tau_D} + \left(\frac{d(R_H(t))}{d\ \tau_D} h_i(t) + \frac{d(h_i(t))}{d\ \tau_D} R_H(t) - R_H(t) h_i(t)\right)$$

Korzystając z (Z26) i (Z38):

$$\frac{d(\varepsilon_i(t))}{d\ \tau_D} = \frac{d\left(\mu(t)^{1/(1-\gamma*\varphi)}\right)}{d\ \tau_D} = \frac{\mu(t)^{(1/(1-\gamma*\varphi))-1}}{(1-\gamma*\varphi)} * \frac{d(\mu(t))}{d\ \tau_D} = \tag{Z58}$$

$$= e^{(\sigma_H-\sigma_K)*t + \int_0^t \left(R_k(\xi)* + \varphi(\gamma-1)*\frac{(\dot{\tau}(\xi))}{b+\tau(\xi)}\right) d\xi} * \frac{\mu(t)^{(1/(1-\gamma*\varphi))-1}}{(1-\gamma*\varphi)} *$$

$$*\left(\frac{d\left(\int_0^t \left(R_k(\xi) * (1-\tau_D) + \varphi(\gamma-1) * \frac{(t\dot{r}(\xi) + \dot{\tau}(\xi))}{b + tr(\xi) + \tau(\xi)}\right) d\xi\right)}{d\ \tau_D} *\right.$$

$$*\int_t^\infty [\left(\varphi*\gamma*(b+tr(T)+\tau(T))^{(1-\gamma)*\varphi}*R_H(T))*e^{-(\sigma_H-\sigma_K)*T-\int_0^T\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{\tau(\xi)})}{b+\tau(\xi)}\right)d\xi}\right]dT+\varphi\gamma*$$

$$*\int_t^\infty \frac{d\left(\left((b+tr(T)+\tau(T))^{(1-\gamma)\varphi}*R_H(T)(1-\tau_D)\right)e^{(\sigma_K-\sigma_H)T-\int_0^T\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)\frac{(\dot{tr(\xi)}+\dot{\tau(\xi)})}{b+tr(\xi)+\tau(\xi)}\right)d\xi}\right)}{d\ \tau_D}dt)$$

Można zapisać:

$$\frac{d\left(\int_0^t\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr(\xi)}+\dot{\tau(\xi)})}{b+tr(\xi)+\tau(\xi)}\right)d\xi\right)}{d\ \tau_D}= \qquad \text{(Z59)}$$

$$=\int_0^t\left(\frac{d(R_k(\xi))}{d\tau_D}-R_k(\xi)+\frac{1-\gamma}{\varphi*(b+\tau(\xi))^2}*(\frac{d(\dot{tr(\xi)}}{d\ \tau_D}*(b+\tau(\xi))-\frac{d(tr(\xi))}{d\ \tau_D}*\dot{\tau(\xi)}\right)d\xi$$

Zauważając, że $tr(t)=\tau_D*(R_k(t)*K(t)+R_H(t)*H(t))+\tau_C*\int_0^1 c_i(t)di$

otrzymuje się:

$$\frac{d\left(\int_0^t\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr(\xi)}+\dot{\tau(\xi)})}{b+tr(\xi)+\tau(\xi)}\right)d\xi\right)}{d\ \tau_D}= \qquad \text{(Z60)}$$

$$=\int_0^t[\frac{d(R_k(\xi))}{d\tau_D}-R_k(\xi)+\frac{\varphi(\gamma-1)}{(b+\tau(\xi))^2}*$$

$$*(\frac{d\left(\tau_D*\left(\dot{R_k}(\xi)*K(\xi)+R_k(\xi)*\dot{K}(\xi)+\dot{R_H}(\xi)*H(\xi)+R_H(\xi)*\dot{H}(\xi)\right)\right)}{d\ \tau_D}*(b+\tau(\xi))$$

$$-\frac{d(\ \tau_D*(R_k(\xi)*K(\xi)+R_H(\xi)*H(\xi)))}{d\ \tau_D}*\dot{\tau}(t))]d\xi=$$

$$=\int_0^t[\frac{d(R_k(\xi))}{d\tau_D}-R_k(\xi)+\frac{\varphi(\gamma-1)}{(b+\tau(\xi))^2}*$$

$$*((b+\tau(\xi))*\left(\dot{R_k}(\xi)*K(\xi)+R_k(\xi)*\dot{K}(\xi)+\dot{R_H}(\xi)*H(\xi)+R_H(\xi)*\dot{H}(\xi)\right)-$$

$$-\dot{\tau}(\xi)*(R_k(\xi)*K(\xi)+R_H(\xi)*H(\xi)))]d\xi=\int_0^t[\frac{d(R_k(\xi))}{d\tau_D}-R_k(\xi)+$$

$$+\frac{\varphi(\gamma-1)*Y(\xi)}{(b+\tau(\xi))^2}*((b+\tau(\xi))*\frac{\dot{Y}(\xi)}{Y(\xi)}-\dot{\tau}(\xi))]d\xi$$ [9]

Dalej:

$$\int_t^\infty\frac{d\left((b+tr(T)+\tau(T))^{(1-\gamma)\varphi}*R_H(T)(1-\tau_D))*e^{-\int_0^T\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi}\right)}{d\ \tau_D}dT \quad (Z61)$$

$$=\int_t^\infty[e^{-\int_0^T\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi}*(\frac{d((b+tr(T)+\tau(T))^{(1-\gamma)*\varphi}*R_H(T)(1-\tau_D))}{d\tau_D}-$$

$$-\frac{d(\int_0^T\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi)}{d\tau_D}*$$

$$*(b+tr(T)+\tau(T))^{(1-\gamma)*\varphi}*R_H(T)(1-\tau_D)))]dT=$$

$$=\int_t^\infty[e^{-\int_0^T\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{\tau}(\xi))}{b+\tau(\xi)}\right)d\xi}*(b+tr(T)+\tau(T))^{(1-\gamma)*\varphi}*R_H(T)(1-\tau_D)*((1-\gamma)*$$

$$*\varphi*\frac{\frac{d(tr(T))}{d\ \tau_D}}{b+tr(T)+\tau(T)}+$$

$$+\frac{\frac{d(R_H(T))}{d\ \tau_D}}{R_H(T)}-1-\int_0^T\frac{d\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(\dot{tr}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)}{d\ \tau_D}d\xi)]dT$$

Zatem, podstawiając (Z60) i (Z61) pod (Z58), otrzymuje się:

$$\frac{d(\varepsilon_i(t))}{d\ \tau_D}=\frac{d\left(\mu(t)^{1/(1-\gamma*\varphi)}\right)}{d\ \tau_D}=e^{\int_0^t\left(R_k(\xi)*+\varphi(\gamma-1)*\frac{(\dot{\tau}(\xi))}{b+\tau(\xi)}\right)d\xi}*\frac{\mu(t)^{(1/(1-\gamma*\varphi))-1}}{(1-\gamma*\varphi)}* \quad (Z62)$$

---

9 $\frac{\dot{R}_k(t)*K(t)+R_k(t)*\dot{K}(t)}{Y(t)}=\alpha*\left(\frac{\dot{Y}(t)}{Y(t)}-\frac{\dot{K}(t)}{K(t)}+\frac{\dot{K}(t)}{K(t)}\right)=\alpha*\frac{\dot{Y}(t)}{Y(t)}$

$$*\left[\int_0^t\{\frac{d(R_k(\xi))}{d\tau_D}-R_k(\xi)+\frac{\varphi(\gamma-1)*Y(\xi)}{(b+\tau(\xi))^2}*\left[(b+\tau(\xi))*\frac{\dot{Y}(\xi)}{Y(\xi)}-\dot{\tau}(\xi)\right]\}d\xi *\right.$$

$$*\int_t^\infty(\varphi\gamma(b+\tau(T))^{(1-\gamma)*\varphi}*R_H(T))*e^{-\int_0^T\left(R_k(\xi)+\varphi(\gamma-1)*\frac{(\dot{\tau}(\xi))}{b+\tau(\xi)}\right)d\xi}dT+$$

$$+\int_t^\infty\{(\varphi\gamma*e^{-\int_0^T\left(R_k(\xi)+\varphi(\gamma-1)*\frac{(\dot{\tau}(\xi))}{b+\tau(\xi)}\right)d\xi}*((b+\tau(T))^{(1-\gamma)\varphi}*R_H(T))*$$

$$*((1-\gamma)*\varphi*\frac{\frac{d(tr(T))}{d\ \tau_D}}{b+\tau(T)}+\frac{\frac{d(R_H(T))}{d\ \tau_D}}{R_H(T)}-1-\int_0^T(\frac{d(R_k(\kappa))}{d\tau_D}-R_k(\kappa)+$$

$$\left.+\frac{\varphi(\gamma-1)*Y(\kappa)}{(b+\tau(\kappa))^2}*\left((b+\tau(\kappa))*\frac{\dot{Y}(\kappa)}{Y(\kappa)}-\dot{\tau}(\kappa))\right)d\kappa\}dT\right]$$

$$\frac{d(h_i(t))}{d\ \tau_D}=\frac{d(e^{-\sigma_H t}*\left(h(0)+\int_0^t e^{\sigma_H\pi}*\left(\varepsilon_i(\pi)^\gamma*(b+\tau+tr(\pi))^{1-\gamma}\right)^\varphi d\pi\right)}{d\ \tau_D}= \qquad \text{(Z63)}$$

$$=\int_0^t\frac{d\left(e^{\sigma_H\pi}*\left(\varepsilon_i(\pi)^\gamma*(b+\tau+tr(\pi))^{1-\gamma}\right)^\varphi\right)}{d\ \tau_D}d\pi=$$

$$=\int_0^t e^{\sigma_H\pi}\varphi\left(\varepsilon_i(\pi)^\gamma*(b+\tau(\pi))^{1-\gamma}\right)^\varphi*\left(\gamma\frac{\frac{d(\varepsilon_i(\pi))}{d\ \tau_D}}{\varepsilon_i(\pi)}+(1-\gamma)\frac{\frac{d(tr(\pi))}{d\ \tau_D}}{(b+\tau(\pi))}\right)d\pi$$

Znając (Z62) i podstawiając (Z63) pod (Z57), otrzymuje się:

$$\frac{d(e^{\sigma_K t-(1-\tau_D)\int_0^t R_K(q)dq}*((1-\tau_D)*R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t)))}{d\ \tau_D}= \qquad \text{(Z64)}$$

$$=e^{\sigma_K t-\int_0^t R_K(q)dq}*((R_H(t)*h_i(t)-\varepsilon_i(t)-\tau(t))*\left(\int_0^t R_K(q)-\frac{d(R_K(q))}{d\ \tau_D}dq\right)-$$

$$-\frac{d(\varepsilon_i(t))}{d\ \tau_D}+(\left(\frac{d(R_H(t))}{d\ \tau_D}-R_H(t)\right)h_i(t)+\int_0^t e^{\sigma_H\pi}\varphi\left(\varepsilon_i(\pi)^\gamma*(b+\tau(\pi))^{1-\gamma}\right)^\varphi*$$

$$* \left(\gamma \frac{\frac{d(\varepsilon_i(\pi))}{d\ \tau_D}}{\varepsilon_i(\pi)} + (1-\gamma)\frac{Y(\pi)}{(b+\tau(\pi))} \right) d\pi * R_H(t))$$

$$\frac{d\left(e^{\frac{(-\rho)t-(1-\theta)*(\sigma_K t - \int_0^t (R_k(T)*(1-\tau_D))dT}{\theta}}\right)}{d\ \tau_D} = e^{\frac{(-\rho)t-(1-\theta)*(\sigma_K t - \int_0^t (R_k(T)*(1-\tau_D))dT)}{\theta}} * \tag{Z65}$$

$$* \frac{d\left(\frac{(-\rho)t + (1-\theta)*(-\sigma_K t + \int_0^t (R_k(T)*(1-\tau_D))dT}{\theta}\right)}{d\ \tau_D} =$$

$$= e^{\frac{(-\rho)t-(1-\theta)*(\sigma_K t - \int_0^t (R_k(T))dT)}{\theta}} * \frac{(1-\theta)}{\theta} * \frac{d\int_0^t (R_k(T)*(1-\tau_D))dT}{d\ \tau_D} =$$

$$= e^{\frac{(-\rho)t-(1-\theta)*(\sigma_K t - \int_0^t (R_k(T))dT}{\theta}} * \frac{(1-\theta)}{\theta} * \int_0^t \frac{dR_k(T)}{d\ \tau_D} - (R_k(T))dT$$

Podstawiając (Z64) i (Z65) pod (Z56), otrzymuje się:

$$\frac{d\left((c(0))^{1-\theta}\right)}{d\ \tau_D} = \tag{Z66}$$

$$\nu_4 * [\nu_2 * (\int_0^\infty \{e^{\sigma_K t - \int_0^t R_K(q)dq} * ((R_H(t) * h_i(t) - \varepsilon_i(t) - \tau(t)) *$$

$$* \left(\int_0^t R_K(q) - \frac{d(R_K(q))}{d\ \tau_D} dq \right) - \frac{d(\varepsilon_i(t))}{d\ \tau_D} + \left(\left(\frac{d(R_H(t))}{d\ \tau_D} - R_H(t) \right) h_i(t) + \right.$$

$$+ \int_0^t e^{\sigma_H \pi} \varphi\left(\varepsilon_i(\pi)^\gamma * (b+\tau(\pi))^{1-\gamma}\right)^\varphi * \left(\gamma \frac{\frac{d(\varepsilon_i(\pi))}{d\ \tau_D}}{\varepsilon_i(\pi)} + (1-\gamma)\frac{Y(\pi)}{(b+\tau(\pi))} \right) d\pi * R_H(t))\}dt) -$$

$$-\nu_3 * \int_0^\infty \{e^{\frac{(-\rho)t-(1-\theta)*(\sigma_K t - \int_0^t (R_k(T))dT}{\theta}} * \frac{(1-\theta)}{\theta} * \int_0^t \frac{dR_k(T)}{d\ \tau_D} - (R_k(T))dT\}dt]$$

Podstawiając równania (Z52) i (Z66) do równania (Z50), otrzymuje się:

$$\frac{dU(\tau_D,\tau_C)}{d\ \tau_D}=\frac{1}{1-\theta}*\int_0^\infty\{e^{-\rho t}*\{c(0)^{1-\theta}*\Delta(t)*\frac{(1-\theta)}{\theta}* \tag{Z67}$$

$$*\int_0^t\frac{d(R_k(M))}{d\ \tau_D}-R_k(M)dM+\Delta(t)*v_4*$$

$$*[v_2*(\int_0^\infty[e^{\sigma_K Q-\int_0^Q R_K(q)dq}*((R_H(Q)*h_i(Q)-\varepsilon_i(Q)-\tau(Q))*$$

$$*\left(\int_0^Q R_K(q)-\frac{d(R_K(q))}{d\ \tau_D}dq\right)-\frac{d(\varepsilon_i(Q))}{d\ \tau_D}+(\left(\frac{d(R_H(Q))}{d\ \tau_D}-R_H(Q)\right)h_i(Q)+$$

$$+\int_0^Q e^{\sigma_H\pi}\varphi\left(\varepsilon_i(\pi)^\gamma*(b+\tau(\pi))^{1-\gamma}\right)^\varphi*\left(\gamma\frac{\frac{d(\varepsilon_i(\pi))}{d\ \tau_D}}{\varepsilon_i(\pi)}+(1-\gamma)\frac{Y(\pi)}{(b+\tau(\pi))}\right)d\pi]*R_H(Q))]dQ)-$$

$$-v_3*\int_0^\infty\left(e^{\frac{(-\rho)G-(1-\theta)*(\sigma_K G-\int_0^G(R_k(T))dT}{\theta}}*\frac{(1-\theta)}{\theta}*\int_0^G\frac{dR_k(T)}{d\ \tau_D}-(R_k(T))dT)dG\right]\}\}dt$$

Zauważmy:

$$v_1=\frac{v_4*\theta}{(1-\theta)*c(0)^{1-\theta}} \tag{Z68}$$

Otrzymuje się:

$$\frac{dU(\tau_D,\tau_C)}{d\ \tau_D}=\int_0^\infty e^{-\rho t}*\frac{\Delta(t)*c(0)^{1-\theta}}{\theta}*\{\int_0^t\frac{d(R_k(M))}{d\ \tau_D}-R_k(M)dM+v_1* \tag{Z69}$$

$$*[v_2*(\int_0^\infty[e^{\sigma_K Q-\int_0^Q R_K(q)dq}*((R_H(Q)*h_i(Q)-\varepsilon_i(Q)-\tau(Q))*$$

$$*\left(\int_0^Q R_K(q)-\frac{d(R_K(q))}{d\ \tau_D}dq\right)-\frac{d(\varepsilon_i(Q))}{d\ \tau_D}+(\left(\frac{d(R_H(Q))}{d\ \tau_D}-R_H(Q)\right)h_i(Q)+$$

$$+\int_0^Q e^{\sigma_H\pi}\varphi\left(\varepsilon_i(\pi)^\gamma*(b+\tau(\pi))^{1-\gamma}\right)^\varphi*\left(\gamma\frac{\frac{d(\varepsilon_i(\pi))}{d\ \tau_D}}{\varepsilon_i(\pi)}+(1-\gamma)\frac{Y(\pi)}{(b+\tau(\pi))}\right)d\pi*R_H(Q))]dQ)-$$

$$-v_3 * \int_0^{\infty} (e^{\frac{(-\rho)G-(1-\theta)*(\sigma_K G-\int_0^G (R_k(T))dT}{\theta}} * \frac{(1-\theta)}{\theta} * \int_0^t \frac{dR_k(T)}{d\ \tau_D} - (R_k(T))dT)dG]\}\ dt$$

Z (Z6):

$$\frac{d(R_k(t))}{d\ \tau_D} - R_k(t) = \frac{d\left(\frac{\alpha}{K(t)}Y(t)\right)}{d\ \tau_D} - \frac{\alpha}{K(t)}Y(t) = \tag{Z70}$$

$$= \alpha(\frac{d(A(t))}{d\ \tau_D}(K(t))^{\alpha-1} * (H(t))^{1-\alpha} + (\alpha-1)\frac{d(K(t))}{d\ \tau_D} * A(t) * (K(t))^{\alpha-2} * (H(t))^{1-\alpha} +$$

$$+(1-\alpha) * \frac{d(H(t))}{d\ \tau_D} * A(t) * (K(t))^{\alpha-1} * (H(t))^{-\alpha} - A(t) * (K(t))^{\alpha-1} * (H(t))^{1-\alpha}) =$$

$$= R_k(t) * \left(\frac{\frac{d(A(t))}{d\ \tau_D}}{A(t)} + (1-\alpha) * \frac{\frac{d(H(t))}{d\ \tau_D}}{H(t)} + (\alpha-1)\frac{\frac{d(K(t))}{d\ \tau_D}}{K(t)} - 1 \right)$$

Analogicznie:

$$\frac{d(R_H(t))}{d\ \tau_D} - R_H(t) = R_H(t) * \left(\frac{\frac{d(A(H(t)))}{d\ \tau_D}}{A(t)} - \alpha * \frac{\frac{d(H(t))}{d\ \tau_D}}{H(t)} + \alpha\frac{\frac{d(K(t))}{d\ \tau_D}}{K(t)} - 1 \right) \tag{Z71}$$

z równości $\frac{\frac{d(H(t))}{d\ \tau_D}}{H(t)} = \frac{\frac{d(K(t))}{d\ \tau_D}}{K(t)}$:

$$\frac{d(R_k(t))}{d\ \tau_D} - R_k(t) = R_k(t) * \left(\frac{\frac{d(A(t))}{d\ \tau_D}}{A(t)} - 1 \right) \tag{Z72}$$

$$\frac{d(R_H(t))}{d\ \tau_D} - R_H(t) = R_H(t) * \left(\frac{\frac{d(A(H(t)))}{d\ \tau_D}}{A(t)} - 1 \right) \tag{Z73}$$

dostaje się zatem warunki:

1. $$\frac{d(R_k(t))}{d\ \tau_D} - R_k(t) > 0 \Leftrightarrow \frac{\frac{d(A(H(t)))}{d\ \tau_D}}{A(t)} > 1$$

2. $$\frac{d(R_H(t))}{d\ \tau_D} - R_H(t) > 0 \Leftrightarrow \frac{\frac{d(A(H(t)))}{d\ \tau_D}}{A(t)} > 1$$

Z (4):

$$\frac{d(A(H(t)))}{d\ \tau_D} = \frac{d(A(H(t)))}{d(H(t))} * \frac{d(H(t))}{d\ \tau_D} = \tag{Z74}$$

$$\frac{d(A_i(0)e^{\left(\int_0^t (g(H_i(s)) - \Phi(H_i(s))\right)ds}[1 + \frac{T_m(0)}{A_i(0)}(\int_0^t \Phi(H_i(\omega))\left(e^{\int_0^\omega g(H_m(\xi))d\xi}\right)e^{\int_0^\omega (\Phi(H_i(\zeta)) - g(H_i(\zeta)))d\zeta} d\omega)}{d(H(t))} *$$

$$* \frac{d(H(t))}{d\ \tau_D} = e^{\left(\int_0^t (g(H_i(s)) - \Phi(H_i(s))\right)ds} * [A_i(0)\left(\int_0^t (\frac{d(g(H\ (s)))}{d(H(s))} - \frac{d(\Phi(H\ (s)))}{d(H(s))} ds\right) +$$

$$+ T_m(0)\int_0^t (e^{g\omega}) * e^{\int_0^\omega (\Phi(H_i(\zeta)) - g(H_i(\zeta)))d\zeta} * (\frac{d(\Phi(H(\omega)))}{d(H(\omega))} + \Phi(H(\omega)) *$$

$$* \int_0^\omega \left(\frac{d(\Phi(H(\zeta)))}{d(H(\zeta))} - \frac{d(g(H(\zeta)))}{d(H(\zeta))}\right) d\zeta) d\omega] * \frac{d(H(t))}{d\ \tau_D}$$

Rozważmy przypadek, kiedy $\frac{d(g(H\ (t)))}{d(H(t))} = \frac{d(\Phi(H\ (t)))}{d(H(t))}$ dla każdego H(t).

Wówczas:

$$\frac{d(A(H(t)))}{d\ \tau_D} = \frac{d(H(t))}{d\ \tau_D} * e^{\left(\int_0^t (g(H_i(s)) - \Phi(H_i(s))\right)ds} *$$

$$* T_m(0)\int_0^t (e^{g\omega}) * e^{\int_0^\omega (g(H_i(\zeta)) - \Phi(H_i(\zeta)))d\zeta} * \frac{d(\Phi(H(\omega)))}{d(H(\omega))} d\omega \tag{Z75}$$

Korzystając z (Z63)[10]:

$$\frac{d(H(t))}{d\ \tau_D} > 0 \Leftrightarrow \int_0^t e^{\sigma_H \pi} \varphi\left(\varepsilon(\pi)^{\gamma} * (b+\tau(\pi))^{1-\gamma}\right)^{\varphi} * \tag{Z76}$$

$$* \left(\gamma \frac{\frac{d(\varepsilon(\pi))}{d\ \tau_D}}{\varepsilon(\pi)} + (1-\gamma)\frac{Y(\pi)}{(b+\tau(\pi))} \right) d\pi > 0$$

Zatem, aby warunki 1 i 2 zostały spełnione, musi zachodzić:

$$\frac{\frac{d(A(H(t)))}{d\ \tau_D}}{A(t)} > 1 \Leftrightarrow \tag{Z77}$$

$$\frac{\frac{d(H(t))}{d\ \tau_D} * e^{\left(\int_0^t (g(H_i(s)) - \Phi(H_i(s))\right) ds} * T_m(0) \int_0^t (e^{g\omega}) * e^{\int_0^{\omega} (\Phi(H_i(\zeta)) - g(H_i(\zeta))) d\zeta} * \frac{d(\Phi(H(\omega)))}{d(H(\omega))} d\omega}{A_i(0) e^{\left(\int_0^t (g(H_i(s)) - \Phi(H_i(s))\right) ds} [1 + \frac{1}{A_i(0)} \left(\int_0^t \Phi(H_i(\omega)) \left(T_m(0) e^{\int_0^{\omega} g(H_m(\xi)) d\xi} \right) e^{\int_0^{\omega} (\Phi(H_i(\zeta)) - g(H_i(\zeta))) d\zeta} d\omega \right]}$$

$$> 1 \Leftrightarrow \frac{\frac{d(H(t))}{d\ \tau_D} * \int_0^t (e^{g\omega}) * e^{\int_0^{\omega} (\Phi(H_i(\zeta)) - g(H_i(\zeta))) d\zeta} * \frac{d(\Phi(H\ (\omega)))}{d(H(\omega))} d\omega}{\frac{A_i(0)}{T_m(0)} + \int_0^t \Phi(H_i(\omega))(e^{g\omega}) e^{\int_0^{\omega} (\Phi(H_i(\zeta)) - g(H_i(\zeta))) d\zeta} d\omega} > 1 \Leftrightarrow$$

$$\frac{d(H(t))}{d\ \tau_D} * \int_0^t (e^{g\omega}) * e^{\int_0^{\omega} (g(H_i(\zeta)) - \Phi(H_i(\zeta))) d\zeta} * (\frac{d(\Phi(H\ (\omega)))}{d(H(\omega))} - \Phi(H_i(\omega)) d\omega > \frac{A_i(0)}{T_m(0)}$$

Podstawiając $\frac{d(R_k(t))}{d\ \tau_D} - R_k(t) = a(t)$ i $\frac{d(R_H(t))}{d\ \tau_D} - R_H t = f(t)$ do (Z69), otrzymuje się:

$$\frac{dU(\tau_D, \tau_C)}{d\ \tau_D} = \int_0^{\infty} e^{-\rho t} * \frac{\Delta(t) * c(0)^{1-\theta}}{\theta} * \{ \int_0^t a(M) dM + \nu_1 * \tag{Z78}$$

$$* [\nu_2 * (\int_0^{\infty} [e^{\sigma_K Q - \int_0^Q R_K(q) dq} * ((R_H(Q) * h_i(Q) - \varepsilon_i(Q) - \tau(Q)) *$$

10 Wykorzystuje się tu fakt, że w tej gospodarce wszyscy wydają tę samą kwotę na prywatną edukację.

$$*\left(\int_0^Q -a(q)dq\right)-\frac{d(\varepsilon_i(Q))}{d\tau_D}+((f(Q))h_i(Q)+$$

$$+\int_0^Q e^{\sigma_H \pi}\varphi\left(\varepsilon_i(\pi)^{\gamma}*(b+\tau(\pi))^{1-\gamma}\right)^{\varphi}*\left(\gamma\frac{\frac{d(\varepsilon_i(\pi))}{d\ \tau_D}}{\varepsilon_i(\pi)}+(1-\gamma)\frac{Y(\pi)}{(b+\tau(\pi))}\right)d\pi)$$

$$*R_H(Q)]dQ)-v_3*\int_0^\infty\left(e^{\frac{(-\rho)G-(1-\theta)*(\sigma_K G-\int_0^G(R_k(T))dT}{\theta}}*\frac{(1-\theta)}{\theta}*\int_0^G a(T)dT)dG\right]\}dt$$

Korzyści z wprowadzenia podatku dochodowego

Procedura jest analogiczna:

$$\frac{dU(\tau_D,\tau_C)}{d\tau_c}=\frac{1}{1-\theta}* \tag{Z79}$$

$$*\int_0^\infty\{(\frac{d}{d\tau_c}\left(\left(\frac{k_i(0)+\int_0^t e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))d\kappa}{(1+\tau_C)*\int_0^\infty e^{\sigma_K*\varsigma-(1-\tau_D)\int_0^\varsigma R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^\varsigma(R_k(T)*(1-\tau_D))dT}{\theta}}d\varsigma}\right)^{1-\theta}\right)*$$

$$*e^{\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}}+\frac{d\left(e^{\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}}\right)}{d\tau_c}*$$

$$*\left(\frac{k_i(0)+\int_0^t e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))d\kappa}{(1+\tau_C)*\int_0^\infty e^{\sigma_K*\varsigma-(1-\tau_D)\int_0^\varsigma R_K(q)dq}*e^{\frac{(-\sigma_K-\rho)t+\int_0^\varsigma(R_k(T)*(1-\tau_D))dT}{\theta}}d\varsigma}\right)^{1-\theta})*e^{-\rho t}\}dt$$

Podstawiając analogicznie do (Z50), otrzymuje się:

$$\frac{dU(\tau_D,\tau_C)}{d\tau_c}=\frac{1}{1-\theta}*\int_0^\infty e^{-\rho t}*(\frac{d\left(c(0)^{1-\theta}\right)}{d\tau_c}*\Delta(t)+\frac{d(\Delta(t))}{d\tau_c}*c(0)^{1-\theta})dt \tag{Z80}$$

$$\frac{d(\Delta(t))}{d\tau_c}=\frac{d\left(e^{\frac{(1-\theta)*((-\sigma_K-\rho)t+\int_0^t(R_k(M)*(1-\tau_D))dM)}{\theta}}\right)}{d\tau_c}= \tag{Z81}$$

$$=\frac{(1-\theta)*\Delta(t)}{\theta}*\int_0^t\frac{d(R_k(M)*(1-\tau_D))}{d\tau_c}dM$$

Analogicznie:

$$\frac{d((c(0))^{1-\theta})}{d\tau_c} = (c(0))^{-\theta} * (1-\theta) * \frac{d(c(0))}{d\tau_c} = c(0)^{-\theta} * (1-\theta) * \tag{Z82}$$

$$* \frac{d\left(\dfrac{k_i(0) + \int_0^\infty e^{\sigma_K \kappa - (1-\tau_D)\int_0^\kappa R_K(q)dq} * ((1-\tau_D) * R_H(\kappa) * h_i(\kappa) - \varepsilon_i(\kappa) - \tau(\kappa))d\kappa}{(1+\tau_C) * \int_0^\infty e^{\sigma_K * \varsigma - (1-\tau_D)\int_0^\varsigma R_K(q)dq} * e^{\frac{(-\sigma_K - \rho)\varsigma + \int_0^\varsigma (R_k(T)*(1-\tau_D))dT}{\theta}} d\varsigma}\right)}{d\tau_c} =$$

$$= \frac{c(0)^{-\theta} * (1-\theta)}{\left((1+\tau_C) * \int_0^\infty e^{\sigma_K * \varsigma - (1-\tau_D)\int_0^\varsigma R_K(q)dq} * e^{\frac{(-\sigma_K - \rho)\varsigma + \int_0^\varsigma (R_k(T)*(1-\tau_D))dT}{\theta}} d\varsigma\right)^2}$$

$$* (\int_0^\infty \frac{d\left(e^{\sigma_K \kappa - (1-\tau_D)\int_0^\kappa R_K(q)dq} * ((1-\tau_D) * R_H(\kappa) * h_i(\kappa) - \varepsilon_i(\kappa) - \tau(\kappa))\right)}{d\tau_c} d\kappa *$$

$$* (1+\tau_C) * \int_0^\infty e^{\sigma_K * \varsigma - (1-\tau_D)\int_0^\varsigma R_K(q)dq} * e^{\frac{(-\sigma_K - \rho)\varsigma + \int_0^\varsigma (R_k(T)*(1-\tau_D))dT}{\theta}} d\varsigma -$$

$$-(k_i(0) + \int_0^\infty e^{\sigma_K \kappa - (1-\tau_D)\int_0^\kappa R_K(q)dq} * ((1-\tau_D) * R_H(\kappa) * h_i(\kappa) - \varepsilon_i(\kappa) - \tau(\kappa))d\kappa) *$$

$$* (\int_0^\infty e^{\sigma_K * \varsigma - (1-\tau_D)\int_0^\varsigma R_K(q)dq} * e^{\frac{(-\sigma_K - \rho)\varsigma + \int_0^\varsigma (R_k(T)*(1-\tau_D))dT}{\theta}} d\varsigma +$$

$$+ \int_0^\infty \frac{d(e^{\sigma_K * \varsigma - (1-\tau_D)\int_0^\varsigma R_K(q)dq} * e^{\frac{(-\sigma_K - \rho)\varsigma + \int_0^\varsigma (R_k(T)*(1-\tau_D))dT}{\theta}}}{d\tau_c} d\varsigma * (1+\tau_C))$$

Podstawiając (Z55), otrzymuje się:

$$\frac{d\left(c(0)^{1-\theta}\right)}{d\tau_c} = v_4 * (v_2 * \tag{Z83}$$

$$* \int_0^\infty \frac{d\left(e^{\sigma_K \kappa - (1-\tau_D)\int_0^\kappa R_K(q)dq} * ((1-\tau_D) * R_H(\kappa) * h_i(\kappa) - \varepsilon_i(\kappa) - \tau(\kappa))d\kappa\right)}{d\tau_c} - v_3 *$$

$$*\left(\int_0^\infty \frac{d(e^{\sigma_K*\varsigma-(1-\tau_D)\int_0^\varsigma R_K(q)dq} * e^{\frac{(-\sigma_K-\rho)t+\int_0^\varsigma (R_k(T)*(1-\tau_D))dT}{\theta}}}{d\tau_c} d\varsigma*(1+\tau_C)+\nu_2\right))$$

Analogicznie do (Z57):

$$\frac{d\left(e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))\right)}{d\tau_c}= \tag{Z84}$$

$$=e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*(((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))*$$

$$*(\tau_D-1)\int_0^\kappa \frac{d(R_K(q))}{d\tau_c}dq-\frac{d(\varepsilon_i(k))}{d\tau_c}+(1-\tau_D)*R_H(\kappa)*\frac{d(h_i(\kappa))}{d\tau_c}+$$

$$+(1-\tau_D)*\frac{d(R_H(\kappa))}{d\tau_c}*h_i(\kappa))$$

Analogicznie do (Z65):

$$\frac{d(e^{\sigma_K*\varsigma-(1-\tau_D)\int_0^\varsigma R_K(q)dq} * e^{\frac{(-\sigma_K-\rho)\varsigma+\int_0^\varsigma (R_k(T)*(1-\tau_D))dT}{\theta}}}{d\tau_c}= \tag{Z85}$$

$$=\frac{d(e^{\frac{(1-\theta)(-\sigma_K-\rho)\varsigma+\int_0^\varsigma (R_k(T)*(1-\tau_D))dT}{\theta}})}{d\tau_c}=e^{\frac{(1-\theta)*\left((-\sigma_K-\rho)\varsigma+\int_0^\varsigma (R_k(T)*(1-\tau_D))dT\right)}{\theta}}*\frac{(1-\theta)}{\theta}*$$

$$*\int_0^\varsigma \left(\frac{d(R_k(T))}{d\tau_c}*(1-\tau_D)\right)dT$$

Podstawiając (Z84) i (Z85) do (Z83), otrzymuje się:

$$\frac{d\left(c(0)^{1-\theta}\right)}{d\tau_c}=\nu_4*(\nu_2* \tag{Z86}$$

$$*\int_0^\infty [e^{\sigma_K\kappa-(1-\tau_D)\int_0^\kappa R_K(q)dq}*(((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))*$$

$$*(\tau_D-1)\int_0^\kappa (\frac{d(R_K(q))}{d\tau_c}dq-\frac{d(\varepsilon_i(\kappa))}{d\tau_c}+(1-\tau_D)*$$

$$*\left(R_H(\kappa)*\frac{d(h_i(\kappa))}{d\tau_c}+h_i(\kappa)*\frac{d(R_H(\kappa))}{d\tau_c}\right)]d\kappa t-\nu_3*(\nu_2+(1+\tau_C)*$$

$$*\int_0^{\infty}e^{\frac{(1-\theta)*\left((-\sigma_K-\rho)\varsigma+\int_0^{\varsigma}(R_k(T)*(1-\tau_D))dT\right)}{\theta}}*\frac{(1-\theta)}{\theta}*\left(\int_0^{\varsigma}\left(\frac{d(R_k(T))}{d\tau_c}*(1-\tau_D)\right)dT\right)d\varsigma)$$

Podstawiając (Z81) i (Z86) pod (Z80):

$$\frac{dU(\tau_D,\tau_C)}{d\tau_c}=\int_0^{\infty}\{\frac{e^{-\rho t}*\Delta(t)*c(0)^{1-\theta}}{\theta}*(\int_0^{t}\frac{d(R_k(M)*(1-\tau_D))}{d\tau_c}dM)+\nu_1* \qquad (Z87)$$

$$*(\nu_2*\int_0^{\infty}e^{\sigma_K\kappa-(1-\tau_D)\int_0^{\kappa}R_K(q)dq}*(((1-\tau_D)*R_H(\kappa)*h_i(\kappa)-\varepsilon_i(\kappa)-\tau(\kappa))*$$

$$*(\tau_D-1)\int_0^{\kappa}(\frac{d(R_K(q))}{d\ \tau_c}dq-\frac{d(\varepsilon_i(\kappa))}{d\ \tau_c}+(1-\tau_D)*$$

$$*\left(R_H(\kappa)*\frac{d(h_i(\kappa))}{d\tau_c}+h_i(\kappa)*\frac{d(R_H(\kappa))}{d\tau_c}\right)d\kappa-\nu_3*(\nu_2+(1+\tau_C)*$$

$$\int_0^{\infty}e^{\frac{(1-\theta)*\left((-\sigma_K-\rho)t+\int_0^{\varsigma}(R_k(T)*(1-\tau_D))dT\right)}{\theta}}*\frac{(1-\theta)}{\theta}*\left(\int_0^{\varsigma}\left(\frac{d(R_k(T))}{d\ \tau_c}*(1-\tau_D)\right)dT\right)d\varsigma)\}dt$$

$$\frac{d(\varepsilon_i(t))}{d\ \tau_c}=\frac{d\left(\mu(t)^{1/(1-\gamma*\varphi)}\right)}{d\ \tau_c}= \qquad (Z88)$$

$$=\frac{\mu(t)^{(1/(1-\gamma*\varphi))-1}*\varphi*\gamma}{(1-\gamma*\varphi)}*e^{(\sigma_H-\sigma_K)t+\int_0^{t}\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(t\dot{r}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi}*$$

$$*(\int_0^{t}\left(\frac{d(R_k(\xi))}{d\ \tau_c}*(1-\tau_D)+\varphi(\gamma-1)*\frac{d\left(\frac{(t\dot{r}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)}{d\tau_c}\right)d\xi$$

$$*\int_t^{\infty}((b+tr(T)+\tau(T))^{(1-\gamma)*\varphi}*R_H(T)*(1-\tau_D))*$$

$$*e^{-(\sigma_H-\sigma_K)*T-\int_0^{T}\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(t\dot{r}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi}dT+$$

$$+\int_t^\infty [e^{-(\sigma_H-\sigma_K)*T-\int_0^T\left(R_k(\xi)*(1-\tau_D)+\varphi(\gamma-1)*\frac{(t\dot{r}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)d\xi} *$$

$$*(\frac{d((b+tr(T)+\tau(T))^{(1-\gamma)*\varphi}*R_H(T)*(1-\tau_D))}{d\ \tau_c}-$$

$$-\int_0^T\left(\frac{d(R_k(\xi))}{d\ \tau_c}*(1-\tau_D)+\varphi(\gamma-1)*\frac{d\left(\frac{(t\dot{r}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)}{d\ \tau_c}\right)d\xi)]dT)$$

Wiadomo, że $\frac{d(tr(t))}{d\ \tau_c}>0$. Dodatkowo, jako że jest to gospodarka dynamiczna, rosnąca w czasie, więc $\frac{d(t\dot{r}(t))}{d\tau_c}>0$. Więc:

$$\frac{d\left(\frac{(t\dot{r}(\xi)+\dot{\tau}(\xi))}{b+tr(\xi)+\tau(\xi)}\right)}{d\tau_c}=\frac{\frac{d(t\dot{r}(\xi))}{d\ \tau_c}*(b+tr(\xi)+\tau(\xi))-\frac{d(tr(\xi))}{d\ \tau_c}*(t\dot{r}(\xi)+\dot{\tau}(\xi))}{(b+tr(\xi)+\tau(\xi))^2}= \quad (Z89)$$

$$=\frac{(b+\tau(\xi))*\int_0^1 c_i\dot{}(t)di-\dot{\tau}(\xi)*\int_0^1 c_i(\xi)di}{(b+tr(\xi)+\tau(\xi))^2}=$$

$$=\frac{C(\xi)}{(b+tr(\xi)+\tau(\xi))^2}*\left(\frac{R_k(T)-\sigma_K-\rho}{\theta}*(b+\tau(\xi))-\dot{\tau}(\xi)\right)$$

$$\frac{d(h_i(t))}{d\ \tau_c}=\frac{d(e^{-\sigma_H t}*(h(0)+\int_0^t e^{\sigma_H\pi}*\left((\varepsilon_i(\pi))^\gamma*(b+\tau(\pi)+tr(\pi))^{1-\gamma}\right)^\varphi d\pi))}{d\ \tau_c}= \quad (Z90)$$

$$=e^{-\sigma_H t}*\int_0^t\frac{d\left(e^{\sigma_H\pi}*\left((\varepsilon_i(\pi))^\gamma*(b+\tau(\pi)+tr(\pi))^{1-\gamma}\right)^\varphi\right)}{d\ \tau_c}d\pi=$$

$$=e^{-\sigma_H t}\varphi*\int_0^t e^{\sigma_H\pi}*((\varepsilon_i(\pi))^\gamma*(b+\tau(\pi)+tr(\pi))^\gamma)^\varphi*$$

$$*\left(\gamma\frac{\frac{d(\varepsilon_i(\pi))}{d\tau_c}}{(\varepsilon_i(\pi)}+(1-\gamma)\frac{\frac{d(b+\tau+tr(\pi))}{d\tau_c}}{(b+\tau(t)+tr(\pi))}\right)d\pi=$$

$$= e^{-\sigma_H t}\varphi * \int_0^t [e^{\sigma_H \pi} * ((\varepsilon_i(\pi))^{\gamma} * (b+\tau+tr(\pi))^{1-\gamma})^{\varphi} * ((1-\gamma)\frac{C(\pi)}{(b+\tau(\pi))} + \gamma\frac{\frac{d(\varepsilon_i(\pi))}{d\tau_c}}{(\varepsilon_i(\pi)})]d\pi$$

Wyliczamy $\frac{d(R_k(\xi))}{d\ \tau_c}$ analogicznie do (Z72) i (Z73)[11]:

$$\frac{d(R_k(t))}{d\ \tau_c} = R_k(t) * \left(\frac{\frac{d(A(t))}{d\ \tau_c}}{A(t)} + (1-\alpha) * \frac{\frac{d(H(t))}{d\ \tau_c}}{H(t)} - (1-\alpha) * \frac{\frac{d(K(t))}{d\ \tau_c}}{K(t)} \right) = \quad \text{(Z91)}$$

$$= R_k(t) * \frac{\frac{d(A(t))}{d\ \tau_c}}{A(t)} = j(t)$$

$$\frac{d(R_H(t))}{d\ \tau_c} = R_H(t) * \left(\frac{\frac{d(A(t))}{d\ \tau_c}}{A(t)} + (1-\alpha) * \frac{\frac{d(H(t))}{d\ \tau_c}}{H(t)} - (1-\alpha) * \frac{\frac{d(K(t))}{d\ \tau_c}}{K(t)} \right) = \quad \text{(Z92)}$$

$$= R_H(t) * \frac{\frac{d(A(t))}{d\ \tau_c}}{A(t)} = z(t)$$

Analogicznie do (Z74):

$$\frac{d(A(t))}{d\ \tau_c} = \frac{d(A(t))}{d\ H(t)} * \frac{d(H(t))}{d\ \tau_c} \quad \text{(Z93)}$$

Korzystając z (Z86):

$$e^{-\sigma_H t}\varphi * \int_0^t [e^{\sigma_H \pi} * ((\varepsilon_i(\pi))^{\gamma} * (b+\tau(\pi = +tr(\pi))^{1-\gamma})^{\varphi} * ((1-\gamma)\frac{C(\pi)}{(b+\tau(\pi))} + \quad \text{(Z94)}$$

$$+\gamma\frac{\frac{d(\varepsilon_i(\pi))}{d\tau_c}}{(\varepsilon_i(\pi)})]d\pi > 0 \Leftrightarrow$$

$$\int_0^t e^{\sigma_H \pi}((\varepsilon_i(\pi))^{\gamma} * (b+\tau(\pi)+tr(\pi))^{1-\gamma})^{\varphi} * ((1-\gamma)\frac{C(\pi)}{(b+\tau(\pi))} + \gamma\frac{\frac{d(\varepsilon_i(\pi))}{d\tau_c}}{(\varepsilon_i(\pi)})d\pi > 0$$

11 Analogicznie jak w (Z72) równość prywatnych krańcowych korzyści z kapitału i kapitału ludzkiego wymusza, że wychylenia względne w zasobie kapitału i kapitału ludzkiego spowodowane przez transfery muszą być równe.

Zatem, aby $\frac{d(R_k(t))}{d\ \tau_c}$ i $\frac{d(R_H(t))}{d\ \tau_c}$ były większe od 0, musi być spełnione:

$$\frac{\frac{d(A(t))}{d\ \tau_c}}{A(t)} > 0 \Leftrightarrow \frac{\frac{d(H(t))}{d\ \tau_D} * \int_0^t (e^{g\omega}) * e^{\int_0^{\omega}(\Phi(H_i(\zeta)) - g(H_i(\zeta)))d\zeta} * \frac{d(\Phi(H(\omega)))}{d(H(\omega))} d\omega}{\frac{A_i(0)}{T_m(0)} + \int_0^t \Phi(H_i(\omega))(e^{g\omega}) e^{\int_0^{\omega}(\Phi(H_i(\zeta)) - g(H_i(\zeta)))d\zeta} d\omega} > 0 \quad \text{(Z95)}$$

$$\overset{\frac{d(H(t))}{d\ \tau_D} > 0}{\Leftrightarrow} \int_0^t (e^{g\omega}) * e^{\int_0^{\omega}(\Phi(H_i(\zeta)) - g(H_i(\zeta)))d\zeta} * \frac{d(\Phi(H\ (\omega)))}{d(H(\omega))} d\omega > 0$$

Podstawiając (Z90), (Z91) i (Z92) pod (Z87), otrzymuje się:

$$\frac{dU(\tau_D, \tau_C)}{d\ \tau_c} = \int_0^{\infty} \{ \frac{e^{-\rho t} * \Delta(t) * c(0)^{1-\theta}}{\theta} * (\int_0^t j(M)dM) + v_1 * \quad \text{(Z96)}$$

$$* (v_2 * \int_0^{\infty} [e^{\sigma_K \kappa - \int_0^{\kappa} R_K(q)dq} * ((R_H(\kappa) * h_i(\kappa) - \varepsilon_i(\kappa) - \tau(\kappa)) *$$

$$* (-1) \int_0^{\kappa} j(q)dq - \frac{d(\varepsilon_i(\kappa))}{d\ \tau_c} + (R_H(\kappa) * e^{-\sigma_H t} \varphi *$$

$$* \int_0^t e^{\sigma_H \pi} * \varepsilon_i(\pi)^{\gamma - 1} * (b + \tau + tr(\pi))^{-\gamma} * ((1-\gamma)\varepsilon_i(\pi)C(\pi) + \gamma \frac{C(\pi)}{(b + \tau(t) + tr(\pi))} *$$

$$* \left(\frac{R_k(T) - \sigma_K - \rho}{\theta} * (b + \tau(\xi)) - \dot{\tau}(\xi) \right) d\pi + h_i(\kappa) * z(\kappa))] d\kappa -$$

$$- v_3 * (v_2 + \int_0^{\infty} e^{\frac{(1-\theta) * \left((-\sigma_K - \rho)\varsigma + \int_0^{\varsigma} (R_k(T) * (1 - \tau_D))dT \right)}{\theta}} * \frac{(1-\theta)}{\theta} * \left(\int_0^{\varsigma} j(T)dT \right) d\varsigma)\} dt$$